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THEORY OF MODULATION WITH DAMPED WAVES

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ABSTRACT

An electromagnetic wave is modulated for the purpose of transmission of signals by amplitude, frequency or phase modulating methods. It is well known that when a carrier wave is modulated, it gives rise to several side-bands depending on the methods of modulation. The number of side-bands, however, depends on the nature of the modulating wave also. The present paper is concerned with the mathematical study of the effect of modulating a carrier wave with damped sinusoidal modulating wave for amplitude and frequency modulations and with non-periodic variation for phase modulation.

INTRODUCTION

It is well known that an electromagnetic wave can be modulated for the purpose of transmission of signals either by amplitude, frequency or phase modulation. It is further known that when a carrier wave is modulated, it gives rise to a number of sidebands depending on the method of modulation. The effect of modulation, however, has generally been studied for sinusoidal modulating wave. There are several occasions when a carrier wave is modulated with non-sinusoidal waves or non-periodic variations. It is the purpose of this paper to show theoretically the effect of modulating a carrier wave with damped sinusoidal modulating wave and also with non-periodic variations in the case of phase modulations.

1. AMPLITUDE MODULATION

An amplitude-modulated sinusoidal carrier with a sinusoidal modulating wave can be written[1] as a function of time in the form

$$M(t) = A_c(1+\beta_o \cos \omega_r t) \cos(\omega_c t + \phi_c) \qquad (1)$$

This gives rise to two side bands with the carrier wave.

The expression for the modulated carrier with a damped sinusoidal modulating wave can, therefore, be written as

$$M(t) = A_c (1+\beta_0 e^{-\alpha t} \cos \omega_v t) \cos (\omega_c t+\phi_c), \text{ where } \alpha \text{ and } \beta_0 \text{ are constants.} \qquad (2)$$

$$= A_c \cos (\omega_c t+\phi_c) + A_c \beta_0 \left(1-\alpha t + \frac{\alpha^2 t^2}{L2} - \frac{\alpha^3 t^3}{L2} + \ldots \right) \times \cos (\omega_c t+\phi_c).\cos \omega_v t$$

$$= A_c (1+\beta_0 \cos \omega_v t) \cos (\omega_c t + \phi_c)$$

$$- A_c \frac{\beta_0}{2} \left(\alpha t - \frac{\alpha^2 t^2}{L2} + \frac{\alpha\ t^3}{L2} - \ldots \right) \times$$

$$\left[\cos \left\{ (\omega_c+\omega_v) t+\phi_c \right\} \quad \cos \quad \left\{ (\omega_c-\omega_v) t+\phi_c \right\} \right] \quad \ldots (3)$$

It will be observed from expression (3) that, unlike the modulated wave given in expression (1), this modulated wave consists of the original modulated wave and a large number of side bands depending on the duration of the modulating wave.

As a check it may be seen that expression (3) reduces to (1) when $\alpha = o$.

II. Frequency modulation:

The expression for the frequency modulated wave[2,3] with sinusoidal modulation is given by

$$M(t) = A_c \cos \left(\omega_c t + \frac{\Delta \omega_c}{\omega_v} \cos \omega_v t \right). \qquad \ldots (4)$$

$$= A_c \sum_{n=-\infty}^{n=\infty} J_n \left(\frac{\Delta \omega_c}{\omega_v} \right) \cos \left[(\omega_c+n\ \omega_v) t+ \frac{n\pi}{2} \right] \qquad \ldots (5)$$

The expression for the frequency modulated wave with a damped sinusoidal modulating wave may be written as

$$M(t) = A_c \cos (\omega_c t+\beta e^{-\alpha t} \cos \omega_v t), \text{ where } \beta \equiv \frac{\Delta \omega_c}{\omega_v} = \text{modulation index} \qquad \ldots (6)$$

$$= A_c \cos (\omega_c t+\beta_1 \cos \omega_v t), \text{ where } \beta_1 = \beta e^{-\alpha t}$$

$$= A_c \sum_{n=-\infty}^{n=\infty} J_n (\beta_1) \cos \left[(\omega_c+n\ \omega_v) t + \frac{n\pi}{2} \right] \qquad \ldots (6a)$$

Now $J_n(\beta_1) = J_n(\beta e^{-\alpha t}) \equiv F(t)$, say.

It may be noted that when either $\alpha = 0$ or $t = 0$, $\beta_1 = \beta$, i. e. $(\beta_1)_{t=0} = \beta$;

$$\frac{d\beta_1}{dt} = -\alpha\beta e^{-\alpha t}.$$

Expanding F (t) in powers of t, we get

$$F(t) = F(o) + t\left(\frac{dF}{dt}\right)_{t=0} + \frac{t^2}{\underline{|2}}\left(\frac{d^2F}{dt^2}\right)_{t=0} + \cdots,$$

Where $F(o) = J_n(\beta)$,

$$\frac{dF}{dt} = \frac{d\beta_1}{dt} \cdot \frac{d J_n(\beta_1)}{d\beta_1} = -\alpha\beta e^{-\alpha t}\frac{d J_n(\beta_1)}{d\beta_1},$$

$$\text{or}\left(\frac{dF}{dt}\right)_{t=0} = -\alpha\beta\frac{d J_n(\beta)}{d\beta} = -\alpha\beta J_n'(\beta).$$

$$\frac{d^2F}{dt^2} = -\alpha\beta\left(-\alpha e^{-\alpha t}\frac{d J_n(\beta_1)}{d\beta_1} + e^{-\alpha t}\frac{d^2 J_n(\beta_1)}{d\beta_1^2} \cdot \frac{d\beta_1}{dt}\right)$$

$$= -\alpha\beta e^{-\alpha t}\left\{-\alpha\frac{d J_n(\beta_1)}{d\beta_1} - \alpha\beta e^{-\alpha t}\frac{d^2 J_n(\beta_1)}{d\beta_1^2}\right\}$$

$$= \alpha^2\beta e^{-\alpha t}\left\{\frac{d J_n(\beta_1)}{d\beta_1} + \beta e^{-\alpha t}\frac{d^2 J_n(\beta_1)}{d\beta_1^2}\right\}$$

$$\text{or}\left(\frac{d^2F}{dt^2}\right)_{t=0} = \alpha^2\beta\left\{J_n'(\beta) + \beta J_n''(\beta)\right\}$$

Again[4], $J_n'(\beta) = J_{n-1}(\beta) - \frac{n}{\beta}J_n(\beta)$,

and $4J_n''(\beta) = -\frac{4}{\beta}J_n'(\beta) - 4\left(1 - \frac{n^2}{\beta^2}\right)J_n(\beta)$, from Bessel equation

$$= -\frac{4}{\beta^2}\left\{-nJ_n(\beta) + \beta J_{n-1}(\beta)\right\} - 4\left(1 - \frac{n^2}{\beta^2}\right)J_n(\beta)$$

$$= -\frac{4}{\beta}J_{n-1}(\beta) + \frac{4n(n+1)}{\beta^2}J_n(\beta) - 4J_n(\beta)$$

$$= -\frac{4}{\beta} J_{n-1}(\beta) + \frac{2(n+1)}{\beta}\left\{J_{n-1}(\beta) + J_{n+1}(\beta)\right\} - 4 J_n(\beta)$$

$$= \frac{2(n-1)}{\beta} J_{n-1}(\beta) + \frac{2(n+1)}{\beta} J_{n+1}(\beta) - 4 J_n(\beta)$$

$$= \left\{J_{n-2}(\beta) + J_n(\beta)\right\} + \left\{J_n(\beta) + J_{n+2}(\beta)\right\} - 4 J_n(\beta)$$

$$= J_{n-2}(\beta) - 2 J_n(\beta) + J_{n+2}(\beta).$$

Substituting for $J_n'(\beta)$ and $J_n''(\beta)$, we get

$$\left(\frac{d^2F}{d_t^2}\right)_{t=0} = \alpha^2\beta\left[\left\{J_{n-1}(\beta) - \frac{n}{\beta} J_n(\beta)\right\} + \frac{\beta}{4}\left\{J_{n-2}(\beta) - 2 J_n(\beta) + J_{n+2}(\beta)\right\}\right]$$

Thus $J_n(\beta_1) = J_n\left(\beta\, \epsilon^{-\alpha t}\right) = J_n(\beta) - t\left\{\alpha\beta . J_n'(\beta)\right\}$

$$+ \frac{t^2}{\underline{|2}}\left[\alpha^2\beta\left\{J_n'(\beta) + \beta J_n''(\beta)\right\}\right] + \ldots$$

$$= J_n(\beta) - \alpha\beta t\left\{J_{n-1}(\beta) - \frac{n}{\beta} J_n(\beta)\right\}$$

$$+ \alpha^2\beta . \frac{t^2}{\underline{|2}}\left[\left\{J_{n-1}(\beta) - \frac{n}{\beta} J_n(\beta)\right\} + \frac{\beta}{4}\left\}J_{n-2}(\beta) - 2 J_n(\beta) + J_{n+2}(\beta)\right\}\right]$$

$$+ \ldots$$

Substituting the value of $J_n(\beta_1)$ in (6*a*), we get

$$M(t) = A_c \sum_{n=-\infty}^{n=\infty}\left[J_n(\beta) - \alpha\beta t J_n'(\beta) + \alpha^2\beta . \frac{t^2}{\underline{|2}}\left\{J_n'(\beta) + \beta J_n''(\beta)\right\} + \ldots\right] \times \cos\left\{(\omega_c + n\,\omega_r)\,t + \frac{n\pi}{2}\right\} \quad \ldots (7)$$

$$= A_c \sum_{n=-\infty}^{n=\infty} J_n(\beta)\cos\left\{(\omega_c + n\,\omega_r)\,t + \frac{n\pi}{2}\right\}$$

$$+ A_c \sum_{n=-\infty}^{n=\infty}\left[\left\{n J_n(\beta) - \beta J_{n-1}(\beta)\right\}\alpha t \cos\left\{(\omega_c + n\,\omega_r)\,t + \frac{n\pi}{2}\right\}\right]$$

$$+ A_c \sum_{n=-\infty}^{n=\infty} \left[\left\{ J_{n-1}(\beta) - \frac{n}{\beta} J_n(\beta) + \frac{\beta}{4} \cdot \overline{J_{n-2}(\beta) - 2J_n(\beta) + J_{n+2}(\beta)} \right\} \times \alpha^2\beta \cdot \frac{t^2}{\lfloor 2} \cos \left\{ (\omega_c + n\,\omega_r)\,t + \frac{n\pi}{2} \right\} \right] \quad \ldots (8)$$

$+ \ldots$

If the modulation is for a short duration, as is generally found in the case of noise and other disturbances, the terms containing t^2 and higher powers of t may be neglected, and thus expression (8) reduces to

$$M(t) = A_c \sum_{n=-\infty}^{n=\infty} \left[(1 + n\,\alpha t)\,J_n(\beta) - \alpha\beta t\,J_{n-1}(\beta) \right] \cos \left\{ (\omega_c + n\,\omega_r)\,t + \frac{n\pi}{2} \right\} \ldots (9)$$

It is interesting to consider the case when $n = 1$. Thus the amplitude of the modulated wave and its side-bands turns out to be

$$(1 + \alpha t)\,J_1(\beta) - \alpha\beta t\,J_0(\beta).$$

It may be mentioned that this expression may be very small for certain values of β. When α is very large such that αt approximates to 1, we may write the above expression as

$$2\,J_1(\beta) - \beta\,J_0(\beta) \approx 0 \qquad \ldots (10)$$

From the table[5] of Bessel functions, we get $\beta = 8{\cdot}417$ approximately.

When $n = -1$, the corresponding expression is

$$(1 - \alpha t)\,J_{-1}(\beta) - \alpha\beta t\,J_{-2}(\beta)$$
$$= -(1 - \alpha t)\,J_1(\beta) - \alpha\beta t\,J_2(\beta).$$

With the same conditions mentioned above, we get

$$\beta\,J_2(\beta) \approx 0,$$

$$\text{or } \beta \left\{ \frac{2}{\beta} J_1(\beta) - J_0(\beta) \right\} \approx 0, \text{ Since } J_{n+1}(\beta) + J_{n-1}(\beta) = \frac{2n}{\beta} J_n(\beta),$$

or $2\,J_1(\beta) - \beta\,J_0(\beta) \approx 0$, which is the same as equation (10)

Thus it will be seen that, for the modulation index of the order of 8·417, the side-bands, for $n = \pm 1$, almost vanish.

It will be worth-while if we mention that for a general case the amplitude of the side-bands from expression (7) can be written as $J_n(\beta) - \alpha\beta t\,J_n'(\beta)$ approximately.

If this is taken as very small, we get an approximate equation

$$J_n(\beta) - \alpha\beta t\, J_n'(\beta) \approx 0, \quad \text{... (11)}$$

$$\text{or } \alpha t.\ \frac{J_n'(\beta)}{J_n(\beta)} - \frac{1}{\beta} \approx 0$$

$$\text{or } \alpha t \log \left\{ J_n(\beta) \right\} \approx \log (k\beta),$$

$$\text{or } \left\{ J_n(\beta) \right\}^{\alpha t} \approx k\beta, \text{ where } k \text{ is an arbitary constant.} \quad \text{... (12)}$$

We might as well use the recurrence formula of Bessel functions in solving the approximate equation (11).

III. Phase Modulation:

In the case of phase modulation the expression[6,7] for the phase modulated wave with sinusoidal variation is given by

$$M(t) = A_c \cos (\omega_c t + \phi_v \cos \omega_v t) \quad \text{... (13)}$$

$$= A_c \sum_{n=-\infty}^{n=\infty} J_n(\phi_v) \cos \left[(\omega_c + n\,\omega_v) t + \frac{n\pi}{2} \right] \quad \text{... (14)}$$

If however, it is modulated by damped sinusoidal variation, we get the same result as obtained in the case of frequency modulation given in expression (6). Therefore, we shall find the expression of the phase modulated carrier when the phase changes non-sinusoidally with a decaying component. Thus in this particular case, we may write the expression for the phase modulated wave as

$$M(t) = A_c \cos (\omega_c t + \beta e^{-\alpha t}), \text{ where } \beta \text{ is a constant} \quad \text{... (15)}$$

$$= A_c \left\{ \cos \omega_c t \cos (\beta e^{-\alpha t}) - \sin \omega_c t \sin (\beta e^{-\alpha t}) \right\}$$

$$= A_c \cos \omega_c t . \left[\cos \beta + \alpha\beta \sin \beta . t - \alpha^2\beta (\beta \cos \beta + \sin \beta) \frac{t^2}{\lfloor 2} + \alpha^3\beta \left\{ 3\beta \cos \beta - (\beta^2 - 1) \sin \beta \right\} \frac{t^3}{\lfloor 3} + \dots \right]$$

$$- A_c \sin \omega_c t \left[\sin \beta - \alpha\beta \cos \beta . t - \alpha^2\beta (\beta \sin \beta - \cos \beta) \frac{t^2}{\lfloor 2} + \alpha^3\beta \left\{ 3\beta \sin \beta + (\beta^2 - 1) \cos \beta \right\} \frac{t^3}{\lfloor 3} + \dots \right]$$

$$= A_c \left[\cos(\omega_c t+\beta) + \alpha\beta t \sin(\omega_c t+\beta) - \alpha^2\beta \frac{t^2}{\lfloor 2} \times \right.$$

$$\left\{ \beta \cos(\omega_c t+\beta) + \sin(\omega_c t+\beta) \right\}$$

$$\left. + \alpha^3\beta \frac{t^3}{\lfloor 3} \left\{ 3\beta \cos(\omega_c t+\beta) - (\beta^2-1) \sin(\omega_c t+\beta) \right\} + \ldots \right] \qquad \ldots (16)$$

It will be observed from expression (16), that in the case of such modulation, when the duration of modulation is short, and we retain terms upto t^2, we get the expression for the modulated wave as

$$M(t) = A_c \cos(\omega_c t+\beta) - A_c \alpha\beta t \cos\left(\omega_c t+\beta+\frac{\pi}{2}\right) + A_c \alpha^2\beta \sqrt{\beta^2+1}.\frac{t^2}{\lfloor 2}$$

$$\cos\left(\omega_c t+\beta+\psi+\frac{\pi}{2}\right),$$

$$\text{where } \sin\psi = \frac{\beta}{\sqrt{\beta^2+1}} \text{ and } \cos\psi = \frac{1}{\sqrt{\beta^2+1}} \qquad \ldots (17)$$

Thus it will be seen that there will be less number of side-bands compared to the side-bands given in expression (14).

SUMMARY OF RESULTS

I. *Amplitude Modulation*:

If we look to the results as given in expression (3), it will be found that, for modulation with damped sinusoidal wave, in addition to the original wave there is a large number of side-bands depending on the duration of the modulating wave.

II. *Frequency Modulation*:

The remarkable feature of such modulation, for short duration, with damped sinusoidal wave is that, for the modulation index of the order of 8·417, the sidebands corresponding to $n = \pm 1$ almost vanish when $\alpha t \approx 1$. The method has been indicated for finding the condition when the amplitude of side bands may vanish in the general case.

III. *Phase Modulation*:

In case of phase changing non-sinusoidally with a decaying component, the expression for the phase modulated wave, with terms upto t^2, has been obtained and it has been shown that compared to the case of phase modulated wave with sinusoidal variation, there will be much less number of sidebands for such modulation.

My grateful thanks are due to Dr. S. S. Banerjee for his very valuable suggestions and at whose instance this problem was taken up.

REFERENCES

1. Black, H. S., "Modulation Theory", p. 125, D. Van Nostrand Company, Inc. New York (1953).

2. Ibid, p. 191.

3. Ibid, p. 192.

4. Forsyth, A. R., "A Treatise on Differential Equations", p. 180, 4th Edition, Macmillan & Co., London (1914).

5. Korn, G. A. and T. M., "Mathematical Handbook for Scientists and Engineers, pp. 877-878, McGraw-Hill Book Company, Inc. New York (1961).

6. Black, H. S., "Modulatiou Theory", p. 183, D. Van Nostrand Company, Inc. New York (1953).

7. Ibid, p. 188.

ON SOME INFINITE INTEGRALS

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ABSTRACT

In this paper we have proved two theorems on well-known Laplace transform and a few integrals have been evaluated with the application of those theorems. Some of the results given earlier by Saxena (1961) follow as their particular cases.

INTRODUCTION

A function $\phi(p)$ defined by

$$\phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt \tag{1}$$

is called Laplace transform of $h(t)$, which is called its original. Throughout this note (1) shall be represented symbolically as

$$\phi(p) \doteqdot h(t) \tag{2}$$

The object of this paper is to prove two theorems on Laplace transform and to evaluate some infinite integrals by making use of those theorems. The results obtained are quite general and include as particular cases, certain theorems given earlier by Saxena, Sharma and others.

2. **Theorem 1.**† If $\phi(p) \doteqdot h(t)$

and $\psi(p) \doteqdot t^{2\lambda} K_\nu\left(\frac{b}{t}\right) h(t),$

then
$$\psi(p) = \sqrt{\pi}\, 2^{2\lambda} p \int_0^\infty t^{-2\lambda-1} (t+p)^{-1}$$
$$\times\, S_2\left(\frac{\nu-1}{2}, \frac{-\nu-1}{2}, \lambda+\tfrac{1}{2}, \lambda; \frac{bt}{4}\right) \phi(p+t)\, dt \tag{3}$$

provided that the integral is convergent and the Laplace transforms of $|h(t)|$ and $\left| t^{2\lambda} K_\nu\left(\frac{b}{t}\right) h(t)\right|$ exist, $|\arg p| < \pi$, $b > 0$.

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†A similar theorem was also proved by Saxena, R. K.

Proof:— Since $h(t) \doteqdot \phi(p)$

therefore
$$e^{-at} h(t) \doteqdot \frac{p}{p+a} \phi(p+a) \tag{4}$$

by virtue of a well known property.

Also from [Erdelyi, 1954*a* p. 282 (32)]

$$p^{1+2\lambda} \mathrm{K}_\nu \left(\frac{b}{p} \right) \doteqdot \sqrt{\pi}\, 2^{2\lambda}\, t^{-1-2\lambda}\, \mathrm{S}_2 \left(\frac{\nu-1}{2}, \frac{-\nu-1}{2}, \lambda+\tfrac{1}{2}, \lambda; \frac{bt}{4} \right) \tag{5}$$

where $R(2\lambda \pm \nu) < 0$, $b > 0$.

Using (4) and (5) in the Parseval Goldstein theorem [1932, p. 105] of the Operational Calculus, which states that if

$$\phi_1(p) \doteqdot h_1(t) \text{ and } \phi_2(p) \doteqdot h_2(t),$$

then
$$\int_0^\infty \phi_2(t)\, h_1(t)\, t^{-1}\, dt = \int_0^\infty \phi_1(t)\, h_2(t)\, t^{-1}\, dt, \tag{6}$$

we get

$$\int_0^\infty e^{-at} t^{2\lambda} \mathrm{K}_\nu \left(\frac{b}{t} \right) h(t)\, dt = \sqrt{\pi} 2^{2\lambda} \int_0^\infty t^{-2\lambda-1} (t+a)^{-1}$$

$$\times\ \mathrm{S}_2 \left(\frac{\nu-1}{2}, \frac{-\nu-1}{2}, \lambda+\tfrac{1}{2}, \nu; \frac{bt}{4} \right) \phi(t+a)\, dt.$$

Multiplying both sides by a and changing a to p, we arrive at the result stated.

Cor. 1. On taking $\nu = \frac{1}{2}$, we get a theorem recently given by me [1964]. We shall now use the theorem to evaluate two infinite integrals.

Example I. Taking [Saxena, 1960 p. 402 (11)]

$$h(t) = t^\rho\, \mathrm{K}_\mu \left(\frac{b}{t} \right)$$

$$\doteqdot \frac{1}{b\sqrt{\pi}} \left(\frac{2}{p} \right)^{1+\rho} \mathrm{S}_4 \left(\frac{1+\rho}{2}, 1+\frac{\rho}{2}, \frac{\mu}{2}, -\frac{\mu}{2}; \frac{bp}{4} \right)$$

$$= \phi(p)$$

where $R(p) > 0$ and $R(b) > 0$,

therefore

$$t^{2\lambda} \mathrm{K}_\nu \left(\frac{b}{t} \right) h(t) = t^{\rho+2\lambda}\, \mathrm{K}_\nu \left(\frac{b}{t} \right) \mathrm{K}_\mu \left(\frac{b}{t} \right)$$

$$\doteqdot \frac{2^{2\lambda+\rho-1}}{p^{2\lambda+\rho}} G^{60}_{26}\left(\frac{b^2p^2}{4} \middle| \begin{matrix} \Delta(2;o) \\ \Delta(2;1+\rho+2\lambda), \pm\left(\frac{\mu+\nu}{2}\right), \pm\left(\frac{\mu-\nu}{2}\right) \end{matrix} \right)$$

$$= \psi(p)$$

where $R(p) > 0$ and $R(b) > 0$.

Using these values of $\phi(p)$ and $\psi(p)$ in (3), it is found that

$$\int_0^\infty t^{-2\lambda-1}(p+t)^{-\rho-2} S_2\left(\frac{\nu-1}{2}, \frac{-\nu-1}{2}, \lambda+\tfrac{1}{2}, \lambda; \frac{bt}{4}\right)$$

$$\times S_4\left(\frac{1+\rho}{2}, 1+\frac{\rho}{2}, \frac{\mu}{2}, -\frac{\mu}{2}; \frac{b(p+t)}{4}\right) dt$$

$$= \frac{b}{4\,p^{1+\rho+2\lambda}} G^{60}_{26}\left(\frac{b^2p^2}{4} \middle| \begin{matrix} \Delta(2;o) \\ \Delta(2;1+\rho+2\lambda), \pm\left(\frac{\mu+\nu}{2}\right), \pm\left(\frac{\mu-\nu}{2}\right) \end{matrix} \right) \quad (7)$$

where $|\arg p| < \pi$, $b>0$ and $R(2\lambda\pm\nu) < 0$.

Particular case:— If we take $\mu = \frac{1}{2}$, we get a result recently obtained by Saxena [1961, p. 55 (60)].

Example II. Next taking [Erdelyi, 1954*b* p. 142]

$$h(t) = t^{\rho-3/2} J_\nu\left(\frac{b}{t}\right)$$

$$\doteqdot \frac{2^{\rho+\frac{1}{2}}}{\sqrt{\pi}\, b\, p^{\rho-\frac{1}{2}}} S_3\left(\frac{\nu}{2}, \frac{\rho}{2}+\tfrac{1}{4}, \frac{\rho}{2}-\tfrac{1}{4}, -\frac{\nu}{2}; \frac{bp}{4}\right)$$

$$= \phi(p)$$

where $b>0$, $R(\rho) > -1$, $R(p) > 0$.

Also [Saxena, 1960 p. 402 (11)]

$$t^{2\lambda} K_\nu\left(\frac{b}{t}\right) h(t) = t^{2\lambda+\rho-3/2} K_\nu\left(\frac{b}{t}\right) J_\nu\left(\frac{b}{t}\right)$$

$$\doteqdot \frac{2^{2\rho+4\lambda-11/2}}{\pi^2 p^{2\lambda+\rho-3/2}} G^{70}_{08}\left(\frac{b^4p^4}{4^7} \middle| \Delta(4; 2\lambda+\rho-\tfrac{1}{2}), \frac{\nu}{2}, o, \tfrac{1}{2}, -\frac{\nu}{2} \right)$$

$$= \psi(p)$$

where $|\arg b^2| < \pi$ and $R(p) > 0$.

Applying (3) to the above correspondences, we obtain

$$\int_0^\infty t^{-2\lambda-1}(t+p)^{-\rho-\frac{1}{2}} S_2\left(\frac{\nu-1}{2}, \frac{-\nu-1}{2}, \lambda+\tfrac{1}{2}, \lambda; \frac{bt}{4}\right)$$

$$\times S_3\left(\frac{\nu}{2}, \frac{\rho}{2}+\tfrac{1}{4}, \frac{\rho}{2}-\tfrac{1}{4}, \frac{-\nu}{2}; \frac{b(p+t)}{4}\right) dt$$

$$= \frac{b 2^{2\lambda+\rho-6}}{\pi^2 p^{2\lambda}} G^{70}_{08}\left(\frac{b^4p^4}{4^7}\,\middle|\, \Delta(4; 2\lambda+\rho-\tfrac{1}{2}), \frac{\nu}{2}, 0, \tfrac{1}{2}, -\frac{\nu}{2}\right) \tag{8}$$

where $|\arg p| < \pi$, $b > 0$ and $R(2\lambda \pm \nu) > 0$.

Particular case:— On taking $\rho = \frac{1}{2}$, we get

$$\int_0^\infty t^{-2\lambda-1} S_2\left(\frac{\nu-1}{2}, \frac{-\nu-1}{2}, \lambda+\tfrac{1}{2}, \lambda; \frac{bt}{4}\right) K_\nu\left(\sqrt{2b(p+t)}\right)$$

$$J_\nu\left(\sqrt{2b(p+t)}\right) dt$$

$$= \frac{2^{2\lambda-4}}{\pi^{5/2} p^{2\lambda}} G^{70}_{08}\left(\frac{b^4p^4}{4^7}\,\middle|\, \Delta(4; 2\lambda), \frac{\nu}{2}, 0, \tfrac{1}{2}, -\frac{\nu}{2}\right) \tag{9}$$

$|\arg p| < \pi$, $b > 0$ and $R(2\lambda \pm \nu) < 0$.

Theorem II. If $\phi(p) \doteqdot h(t)$ (10)

and $\psi(p) \doteqdot t^{\rho-1} K_\nu\left(\frac{b}{t}\right)\phi(t)$,

then
$$\psi(p) = \frac{p.\, 2^{\rho+1}}{\sqrt{\pi b}} \int_0^\infty (p+t)^{-\rho-2}$$

$$\times S_4\left(\frac{1+\rho}{2}, 1+\frac{\rho}{2}, \frac{\nu}{2}, -\frac{\nu}{2}; \frac{b(p+t)}{4}\right) h(t)\, dt \tag{11}$$

provided that the integral is convergent and the Laplace transforms of $|h(t)|$ and $\left|t^{\rho-1} K_\nu\left(\frac{b}{t}\right)\phi(t)\right|$ exist, $|\arg p| < \pi$, $R(b) > 0$.

Proof:— From [Saxena, 1960 p. 402(11)]

$$e^{-at}\, t^{\rho}\, K_{\nu}\left(\frac{b}{t}\right) \doteqdot \frac{p.\, 2^{\rho+1}}{\sqrt{\pi b}} \frac{1}{(p+a)^{\rho+2}}$$

$$\times S_4\left(\frac{1+\rho}{2}, 1+\frac{\rho}{2}, \frac{\nu}{2}, -\frac{\nu}{2}; \frac{b\,(p+a)}{4}\right) \tag{12}$$

where $R\,(p+a) > 0$ and $R\,(b) > 0$,

on using a well known property.

Apply (6) to (10) and (12), we obtain

$$\int_0^\infty e^{-at}\, t^{\rho-1}\, K_{\nu}\left(\frac{b}{t}\right)\phi\,(t)\, dt = \frac{2^{\rho+1}}{\sqrt{\pi b}} \int_0^\infty (t+a)^{-\rho-2}$$

$$\times\ S_4\left(\frac{1+\rho}{2}, 1+\frac{\rho}{2}, \frac{\nu}{2}, -\frac{\nu}{2}; \frac{b\,(t+a)}{4}\right) h\,(t)\, dt$$

Multiplying both sides by a and changing a to p, we get the result stated.

Cor. I. On taking $\nu=\frac{1}{2}$, we get a result given by [Sharma, 1962]

Now we shall use the theorem to evaluate an infinite integral.

Example 1. Taking [Erdelyi, 1954*a* p. 279]

$$\phi\,(p) = p^{1-\lambda}\, I_{\mu}\left(\frac{b}{p}\right)$$

$$\doteqdot \frac{b^{\mu}}{2^{\mu}} \frac{t^{\lambda+\mu-1}}{\Gamma\,(\mu+1)\,\Gamma\,(\mu+\lambda)}\, {}_0F_3\left(\mu+1, \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; \frac{b^2t^2}{16}\right)$$

$$= h\,(t)$$

where $R\,(\lambda+\mu) > 0$

Also [Saxena, 1960 p. 402 (11)]

$$t^{\rho-1}\, K_{\nu}\left(\frac{b}{t}\right)\phi\,(t) = t^{\rho-\lambda}\, K_{\nu}\left(\frac{b}{t}\right) I_{\mu}\left(\frac{b}{t}\right)$$

$$\doteqdot \frac{2^{\rho-\lambda-1}}{\pi\, p^{\rho-\lambda}}\, G^{42}_{26}\left(\frac{b^2p^2}{4}\,\middle|\, \begin{matrix}\Delta\,(2;\, o) \\ \Delta\,(2;\, 1+\rho-\lambda), \frac{1}{2}\,(\nu+\mu), \frac{1}{2}\,(\mu-\nu), \frac{1}{2}\,(\nu-\mu), -\frac{1}{2}(\nu+\mu)\end{matrix}\right)$$

$$= \psi\,(p)$$

where $|\arg b| < \pi$, $R\,(\rho-\lambda) > 0$ and $R\,(p) > 0$.

Using the above values of $h(t)$ and $\psi(p)$ in (11), we obtain

$$\int_0^\infty t^{\lambda+\mu-1}(p+t)^{-\rho-2}\;{}_0F_3\left(\mu+1,\frac{\lambda+\mu}{2},\frac{\lambda+\mu+1}{2};\frac{b^2t^2}{16}\right)$$

$$\times\; S_4\left(\frac{1+\rho}{2},1+\frac{\rho}{2},\frac{\nu}{2},-\frac{\nu}{2};\frac{b(p+t)}{4}\right)dt$$

$$=\frac{\Gamma(\mu+1)\,\Gamma(\mu+\lambda)\,2^{\mu-\lambda-2}}{\sqrt{\pi}\,p^{1+\rho-\lambda}\,b^{\mu-1}}\,G^{42}_{26}\left(\frac{b^2p^2}{4}\middle|\begin{matrix}\Delta(2;o)\\ \Delta(2;1+\rho-\lambda),\tfrac{1}{2}(\mu\pm\nu),\;-\tfrac{1}{2}(\mu\mp\nu)\end{matrix}\right) \tag{13}$$

where $|\arg p|<\pi$, $b>0$ and $R(\lambda+\mu)>0$.

In particular, if we take $\nu=\frac{1}{2}$, in (13), we get a result recently given by [Saxena, 1961 p. 54 (57)].

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REFERENCES

1. Erdelyi, A., Tables of Integral Transforms, Vol. I, McGraw Hill, New York (1954*a*).
2. Erdelyi, A., Tables of Integral Transforms, Vol. II, McGraw Hill, New York (1954*b*).
3. Gokhroo, D. C., On Hankel transform of certain functions communicated for publication (1964).
4. Goldstein, S., Operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder functions, *Proc. London Math. Soc.*, (2) *34*, p. 103-125 (1932).
5. Saxena, R. K., Some theorems on generalized Laplace transforms-I, *Proc, Nat. Inst. Sci. India*, Vol. *26*A, No. 4, p. 400-413 (1960).
6. Saxena, R. K., Some theorems in Operational Calculus and infinite integrals involving Bessel function and G functions, *Proc. Nat. Inst. Sci. India*, Vol. *27A*, No. 1, p. 38-61 (1961).
7. Sharma, K. C., Thesis approved for Ph. D. degree of Raj. University (1962).

ON THE THEORETICAL CONSIDERATION OF TEMPERATURE DISTRIBUTION IN THE ROUND LAMINAR JET

By

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SUMMARY

In the present paper, I have discussed the temperature distribution for the viscous incompressible flow through the round luminar jet, on the assumption that $\psi = \nu\, r^n f(\theta)$ and $T = r^m g(\theta)$. For $m = -1$, it has been shown that the expression for the temperature distribution becomes similar to the equation of Squire (ref. 2). As assumed in ref. 2, I have neglected the dissipation term due to friction as a small quantity. The expression for the temperature distribution has been calculated when $n=4$, $m = -4$, and the solution is obstained in series.

INTRODUCTION

An exact solution of the Navier-Stokes equations for the viscous incompressible flow in axially—symmetric motion was discussed by Squire (2). The effect of a source of heat at the point of application of the force has also been considered by him. A special form of stream function $\psi = \nu\, r\, f(\theta)$ and temperature distribution $T = \frac{1}{r}\, g(\theta)$ was assumed by Squire. Schlichting (ref. 1, section 57) studied this type of flow using the approximations of the boundary-layer theory. Agarwal (3) made calculations for the stream function for such flow on the assumption that $\psi = \nu\, r^n f(\theta)$ and it was shown by him that $n = 4$ admits an exact solution. However he did not touch the discussion for the temperature distribution. In this paper, I have studied the distribution of temperature for the laminar viscous incompressible flow through the round jet.

SOLUTION OF THE PROBLEM

The equation giving temperature (ref. 4, pp. 72-73) for the steady laminar viscous incompressible flow is

$$q_r \frac{\partial T}{\partial r} + \frac{q_\theta}{r} \frac{\partial T}{\partial \theta} = \frac{k}{\rho c_v} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{r^2 \partial T}{\partial r} \right) + \frac{1}{r^2 \operatorname{Sin}\theta} \frac{\partial}{\partial \theta} \left(\operatorname{Sin}\theta \frac{\partial T}{\partial \theta} \right) \right] + \phi \qquad \text{— (1)}$$

where ϕ is the dissipation function (ref. 4, equs. 2.43 & 2.44) and is defined by

$$\phi = \mu \left[2\left(e_{rr}^2 + e_{\theta\theta}^2 + e_{\phi\phi}^2 \right) + e_{\theta\phi}^2 + e_{\phi r}^2 + e_{r\theta}^2 \right]$$

The symbols used are

$$\left.\begin{aligned}
e_{rr} &= \frac{\partial q_r}{\partial r} \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r} \\
e_{\phi\phi} &= \frac{1}{r \operatorname{Sin}\theta}\frac{\partial q_\phi}{\partial \phi} + \frac{q_r}{r} + \frac{q_\theta \operatorname{Cot}\theta}{r} \\
e_{\theta\phi} &= \frac{\operatorname{Sin}\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{q_\phi}{\operatorname{Sin}\theta}\right) + \frac{1}{r\operatorname{Sin}\theta}\frac{\partial q_\theta}{\partial \phi} \\
e_{\theta r} &= r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial \theta}, \quad e_{\phi r} = \frac{1}{r\operatorname{Sin}\theta}\frac{\partial q_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right)
\end{aligned}\right\} \quad \ldots (2)$$

For the present problem, we take

$$\psi = \nu\, r^n f(\theta) \qquad \ldots (3)$$

Therefore

$$\left.\begin{aligned}
q_r &= -\frac{1}{r^2 \operatorname{Sin}\theta}\frac{\partial \psi}{\partial \theta} = -\nu \frac{r^{n-2}}{\operatorname{Sin}\theta} f'(\theta) \\
q_\theta &= \frac{1}{r\operatorname{Sin}\theta}\frac{\partial \psi}{\partial r} = \frac{\nu_n r^{n-2}}{\operatorname{Sin}\theta} f(\theta) \\
q_\phi &= 0
\end{aligned}\right\} \quad \ldots \qquad \ldots (4)$$

Substituting $\mu = \operatorname{Cos}\theta$, we get

$$\operatorname{Sin}\theta = \sqrt{(1-\mu^2)}$$

$$\frac{1}{\operatorname{Sin}\theta}\frac{d}{d\theta} = -\frac{d}{d\mu}\,.$$

From equations (2) and (4), we get after simplification,

$$\phi = \frac{\nu^3 \rho_r^{2(n-3)}}{\operatorname{Sin}^2\theta}\Big[n^2(n^2 - 6n + 4\operatorname{Cot}^2\theta + 9)f^2 - (2n^2 \operatorname{Cot}\theta + 6n \operatorname{Cot}\theta) f f'$$

$$+ (2n^2 - 10n + \operatorname{Cot}^2\theta + 12) f'^2 - 2n(n-3) f f'' + f''^2 - 2 \operatorname{Cot}\theta .\, f' f''\Big] \ldots (5)$$

Equations of set (4) are replaced by

$$\left.\begin{aligned} q_r &= -\nu\, r^{n-2} f'(\mu) \\ q_\theta &= -\nu n \frac{r^{n-2} f(\mu)}{\sqrt{1-\mu^2}} \\ q_\phi &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots (6)$$

For the temperature distribution in the present problem, we assume that

$$T = r^m g(\theta) \quad \ldots (7)$$

Squire (ref. 2, equ. 14) has taken $m = -1$ in equation (7).

From above substitutions, we get from equation (1).

$$r^{m+n-3}\,[-m f'g + n f g'] = k\, r^{m-2}\left[m(m+1) g + \frac{d}{d\mu}\{ g'(1-\mu^2)\} \right] \ldots (8)$$

where ϕ is given by equation (5) ; f and g are functions of μ.

We have assumed that

$$K = \frac{k}{\nu \rho c_v} = \text{constant}$$

In the viscous flow, we know that ν is very small. In equ. (5), ϕ is multiplied by ν^3. Thus in an approximate calculation, we may neglect ϕ in equation (8), in comparison with rest of the terms.

If ϕ is neglected in equation (8), then it is easily solved for $n = 1$.

If $m = -1$, we have result of ref. 2 which has been discussed by Prof. Squire. For $n = 1$, it has been shown in ref. 2, equ. (11), that

$$f = \frac{2(1-\mu^2)}{\alpha+1-\mu} = \frac{2 \,\text{Sin}^2 \theta}{\alpha+1-\text{Cos}\,\theta}$$

where a is an arbitrasy constant of integration.

If $m \neq -1$, then equation (8) gives at $n = 1$

$$\frac{d^2 g}{d\mu^2} + f_1(\mu)\frac{dg}{d\mu} + f_2(\mu)\, g = 0 \quad \ldots (9)$$

where $f_1(\mu) = -\dfrac{2\mu}{1-\mu^2} - \dfrac{2}{K(\alpha+1-\mu)}$

$$f_2(\mu) = m(m+1) + \frac{2m}{K} - \frac{1+\mu^2 - 2\mu(1+a)}{(\alpha+1-\mu)\,2} \quad \ldots (10)$$

Equation (9) can be solved by substitution

$$g(\mu) = w(\mu)\, v(\mu)$$

where

$$w(\mu) = \text{Exp.}\left\{ -\frac{1}{2}\int f_1(\mu)\, d\mu \right\}$$

$$= (1-\mu^2)^{-\frac{1}{2}} (a+1-\mu)^{-\frac{1}{K}},$$

$$\frac{d^2 v}{d\mu^2} + \text{I} v = 0 \tag{11}$$

and

$$\text{I} = f_2 - \tfrac{1}{2}\frac{df_1}{d\mu} - \tfrac{1}{4} f_1^2$$

$$= m(m+1) + \frac{1}{K(a+1-\mu)^2}\left[1 - \frac{1}{K} + 2m\{1+\mu^2 - 2\mu(1+a)\} + \frac{1}{1-\mu^2} \right]$$

$$- \frac{2\mu}{K(1-\mu^2)(a+1-\mu)}$$

The form of I shows that equation (11) cannot be solved in an closed form for $v(\mu)$. Hence exact solution for $g(\mu)$ from equ. 9 cannot be derived and the possible value of m for an exact solution is -1.

For $n \neq 1$, when we substitute for q_r, q_θ and q_ϕ from equation (4) into equations of motion (ref. 4, pages 72), we get complicated equation (ref. 3, equ. 5) viz.

$$r^{n-1}\Big[-2n^2(n-1)\,\text{Cot}\,\theta \,.\, f^2 + n\{2n-3+3\,\text{Cot}^2\theta\} f f' - 3n\,\text{Cot}\,\theta \,.\, f f''$$
$$+ (n-4)\,\text{Cot}\,\theta\, f'^2 - (n-4) f' f'' + n f f''' \Big]$$

$$= \text{Sin}\,\theta \Big[n(n-1)(n-2)(n-3) f - (2n^2-6n+9+3\,\text{Cot}^2\theta)\,\text{Cot}\,\theta . f' +$$
$$(2n^2-6n+8+3\,\text{Cot}^2\theta) f'' - 2\,\text{Cot}\,\theta\, f''' + f^{iv} \Big] \qquad \ldots (12)$$

where f is a function of θ alone.

For $n = 4$, the solution as found in ref. 3, is

$$f(\theta) = \text{Sin}^2\theta \left[\frac{A}{10} + B\, P_3'(\mu) + C\, Q_3'(\mu) \right] \qquad \ldots (13)$$

where P_3' . Q_3' are the derivatives of the Legendre functions P_3 and Q_3 . A, B and C are certain constants.

From equation (8), for $n = 4$, taking $\phi = 0$, was have

$$r^3 \left[- m f' g + 4 f g' \right] = K \left[m (m + 1) g + \frac{d}{d\mu} \left\{ g' (1-\mu^2) \right\} \right] \quad \ldots (14)$$

where $f(\mu)$, is to be substituted from equ. (13).

After making substitutions, we find that in this case $g(\mu)$ satisfies an equation similar to (11) with $f_1(\mu)$ and $f_2(\mu)$ as functions of Legendre co-efficients. In this case there are complications and exact solution cannot be obtained.

However if $m = -4$, at $n = 4$, equation (14) is reduced to

$$\frac{4 r^3}{K} \frac{d}{d\mu} (f \cdot g) - \left[12 g + \frac{d}{d\mu} \cdot \left\{ (1-\mu^2) \frac{dg}{d\mu} \right\} \right] = 0 \quad \ldots (15)$$

Taking a particular case for $f(\mu)$, in equation (13), when $C = 0$, $\frac{A}{B} = 15$, we get (ref. 3).

$$\psi = k r^4 \mu^2 (1-\mu^2) \quad \ldots (16)$$

where k is a constant. This represents a motion with plane boundary. The stream lines are rectangular hyperbolas. Employing (15) and (16), after a few steps of simplifications, we have

$$(1-\mu^2) \frac{d^2 g}{d\mu^2} - \left[C' \mu^2 (1-\mu^2) + 2\mu \right] \frac{dg}{d\mu} - \left[12 - 2 C' \mu (1-2\mu^2) \right] g = 0 \quad \ldots (17)$$

where C' is a certain constant.

Equation (17) can be solved in series by selecting $g(\mu) = \mu^r [a_0 + a_1 \mu + a_2 \mu^2 + a_3 \mu^3 + \quad . \quad . \quad . \quad]$ ... (18)

Substituting (18) into (17), we get for the co-efficients of μ,

$$r = 1, \; a_1 = 0, \; a_2 = -\frac{5}{8} a_0, \; a_3 = \frac{a_0 (1 + 2 C')}{12}, \; a_4 = a_0 .$$

Thus

$$g(\mu) = a_0 \left[\mu - \frac{5}{8} \mu^3 + \frac{1 + 2C'}{12} \mu^4 + \mu^5 + \quad . \quad . \quad . \right] \quad \ldots (19)$$

From (16), we see that

$u = v = 0$ at $\theta = \frac{\pi}{2}$, which represents motion with plane boundary. From (19), we see that $g(0) = 0$ on the plane boundary. So for such particular choice of n, m and ψ, we see that the plane boundary should deliberately be maintained at constant zero temperature.

REFERENCES

1. Goldstein, S. : Modern Development in Fluid-Dynamics (Oxford, 1938), Vol. I.
2. Squire, H. B. : *Quart. J. of Mech. and Appl. Math.*, Vol. IV, Pt. 3 (1951), 321-329.
3. Agarwal, H. L. : *Quart. J. of Mech. and Appl. Math.*, Vol. X, Pt. 1 (1957), 42-44.
4. Pai, S. : Viscous Flow Theory, I—Laminar Flow (INC., 1956).

FINITE BENDING OF PLATES IV

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ABSTSRACT

The method developed in parts I, II, and III has been applied to the problem of bending of a circular plate into a paraboloidal shell. It is shown that the deformation can be maintained by applying a force and a couple to the edge of the shell only.

1. INTRODUCTION

In parts I, II and III [1-3] we considered the problems of isotropic compressible rectangular and circular blocks bent into elliptic and spherical shells respectively. One of the authors [4] considered the case of a circular plate bent into an ellipsoidal shell. In this paper the method has been extended to the problem of a bending of a circular plate into a paraboloidal shell. As in part III [3] the strain tensor has been calculated directly from the metric tensors of the strained and unstrained states of the body without any reference to the displacements. We use Seth's stress-strain relations [5]

$$\sigma_{ij} = \lambda \, I \, \delta_{ij} + 2\mu \, e_{ij} \tag{1.1}$$

2. NOTATION AND FORMULAE

Following the notation of Green and Zerma [6] let x_i be the initial, y_i the final and θ_i the curvilinear coordinates of the strained body, so that

$$x_i = x_i(\theta_1, \theta_2, \theta_3), \; y_i = y_i(\theta_1, \theta_2, \theta_3) \tag{2·1}$$

The metric tensors g_{ij} and G_{ij} for the coordinate system θ_i in the unstrained and states respectively are given by

$$g_{ij} = \frac{\partial x_m}{\partial \theta_i} \frac{\partial x_n}{\partial \theta_j} \delta_{mn}, \; G_{ij} = \frac{\partial y_m}{\partial \theta_i} \frac{\partial y_n}{\partial \theta_j} \delta_{mn} \tag{2·2}$$

Then the state of strain is measured by the symmetric covariant strain tensor

$$2\gamma_{ij} = G_{ij} - g_{ij} \tag{2·3}$$

If T^{ij} is the contravariant stress tensor, refered to θ_i coordinates in the strained state of the body, the stress strain relations (1·1) can be written as

$$T^{ij} = \mu \left(G^{ir} G^{js} + G^{is} G^{jr} + \frac{2\eta}{1-2\eta} G^{ij} G^{rs} \right) \gamma_{rs} \tag{2·4}$$

where η is poisson's ratio, so that $\frac{2\mu\eta}{1-2\eta} = \lambda$.

The equations of equilibrium in the absence of body forces are

$$T^{ij} \,|\, i = o \tag{2·5}$$

where the vertical line denotes covariant partial differentiation with respect to θ_i in the strained body.

When the curvilinear coordinates θ_i are orthogonal the physical components of stress σ_{ij} and strain ϵ_{ij} are given by

$$\sigma_{ij} = \sqrt{G_{ii} G_{jj}}\, T^{ij}, \; \epsilon_{ij} = \sqrt{G^{ii} G^{jj}}\, \gamma_{ij} \tag{2·6}$$

3. BENDING OF A CIRCULAR PLATE INTO A PARABOLOIDAL SHELL

Let the circular plate in the undeformed state be bounded by the planes $x_3 = a_1$, $x_3 = a_2$, $a_2 > a_1$ and the cylinder $x_1^2 + x_2^2 = a^2$. It is then bent symmetrically about z_3 axis into a part of a paraboloidal shell, whose inner and outer boundaries are the paraboloids of revolution obtained by revolving the confocal parabolas,

$$x_3 = \tfrac{1}{2}(\zeta^2 - \eta^2),\; x_1 = \zeta\eta,\; \zeta = \zeta_i,\; i = 1,2. \tag{3·1}$$

about the x_3 axis respectively and the edge $\eta = \alpha$. Let y_i axes coincide with the x_i axes and the curvilinear coordinates θ_i in the deformed state be a system of orthogonal curvilinear coordinates (ζ, η, φ) where φ is the angle between $y_1 y_3$-plane and the plane through a point in space and y_3 axis.
Then

$$y_1 = \zeta\eta \cos\varphi,\; y_2 = \zeta\eta \sin\varphi,\; y_3 = \tfrac{1}{2}(\zeta^2 - \eta^2) \tag{3·2}$$

Since the deformation is symmetric about the x_3 axis, we assume [1]

$$x_3 = f(\zeta),\; (x_1^2 + x_2^2)^{\frac{1}{2}} = F(\eta),\; \tan^{-1}\frac{x_2}{x_1} = \varphi \tag{3·3}$$

then $x_1 = F(\eta)\cos\varphi$, $x_2 = F(\eta)\sin\varphi$, $x_3 = f(\zeta)$ (3·4)

From (2·2), (2·3), (3·2) and (3 4) we get

$$g_{ij} = \begin{bmatrix} f'^2 & 0 & 0 \\ 0 & F'^2 & 0 \\ 0 & 0 & F^2 \end{bmatrix},\; G_{ij} = \begin{bmatrix} \zeta^2+\eta^2 & 0 & 0 \\ 0 & \zeta^2+\eta^2 & 0 \\ 0 & 0 & \zeta^2\eta^2 \end{bmatrix} \tag{3·5}$$

$$2\gamma_{ij} = \begin{bmatrix} \zeta^2+\eta^2-f'^2 & 0 & 0 \\ 0 & \zeta^2+\eta^2-F'^2 & 0 \\ 0 & 0 & \zeta^2\eta^2 - F^2 \end{bmatrix} \tag{3·6}$$

where $f' = \dfrac{df}{d\zeta}$, $F' = \dfrac{dF}{d\eta}$.

From (3·5), (3·6), (2·4) and (2·6) the non-vanishing stresses are given by

$$2\sigma_{11} = 2(\zeta^2+\eta^2)T^{11} = \lambda\left(3 - \frac{f'^2+F'^2}{\zeta^2+\eta^2} - \frac{F^2}{\zeta^2\eta^2}\right) + 2\mu\left(1 - \frac{f'^2}{\zeta^2+\eta^2}\right) \tag{3·7}$$

$$2\sigma_{22} = 2(\zeta^2+\eta^2)T^{22} = \lambda\left(3 - \frac{f'^2+F'^2}{\zeta^2+\eta^2} - \frac{F^2}{\zeta^2+\eta^2}\right) + 2\mu\left(1 - \frac{F'^2}{\zeta^2+\eta^2}\right) \tag{3·8}$$

$$2\sigma_{33} = 2\zeta^2\eta^2 T^{33} = \lambda\left(3 - \frac{f'^2+F'^2}{\zeta^2+\eta^2} - \frac{F^2}{\zeta^2\eta^2}\right) + 2\mu\left(1 - \frac{F^2}{\zeta^2\eta^2}\right) \tag{3·9}$$

From (3·7), (3·8), (3·9) and (2·5) the equations of equilibrium to be satisfied are

$$\frac{\partial T^{11}}{\partial \zeta} + \frac{T^{11}}{\zeta} + \frac{\zeta}{\zeta^2+\eta^2}(3T^{11} - T^{22}) - \frac{\zeta\eta^2}{\zeta^2+\eta^2} T^{33} = 0 \tag{3·10}$$

$$\frac{\partial T^{2}}{\partial \eta} + \frac{T^{22}}{\eta} + \frac{\eta}{\zeta^2+\eta^2}(3T^{22} - T^{11}) - \frac{\eta\zeta^2}{\zeta^2+\eta^2} T^{33} = 0 \tag{3·11}$$

$$\frac{\partial T^{33}}{\partial \varphi} = 0 \tag{3·12}$$

Equation (3·12) is identically satisfied, since T^{23} is independent of φ.

Substituting (3·7) to (3·9) in (3·10) and (3·11) we get

$$\frac{\lambda+2\mu}{2(\lambda+\mu)} \zeta \frac{df'^2}{d\zeta} + \frac{\mu}{\lambda+\mu} f'^2 = \left(\frac{F}{\eta}\right)^2 \frac{\zeta^2+\eta^2}{\zeta^2} + \frac{\zeta^2(f'^2+F'^2)}{\zeta^2+\eta^2} \tag{3·13}$$

$$\frac{\lambda+2\mu}{2(\lambda+\mu)} \eta \frac{dF'^2}{d\eta} + \frac{\mu}{\lambda+\mu} F'^2 = \left(\frac{F}{\eta}\right)^2 \frac{\zeta^2+\eta^2}{\zeta^2} + \frac{\eta^2(f'^2+F'^2)}{\zeta^2+\eta^2}$$

$$- \frac{\lambda}{(\lambda+\mu)} \left(\frac{F}{\eta}\right) \frac{F'(\zeta^2+\eta^2)}{\zeta^2} \tag{3·14}$$

These equations admit a solution [4] only for small values of η and

$$F = A\eta \tag{3·15}$$

Physically this implies that the maximum value η of which is approximately equal to the ratio of the radius of the circular plate in the deformed state to the focal distance of a point on the edge measured in radians is a small quantity. This, however does not imply the deformation to be small. It only means that the region of validity of the solution is limited.

Substituting (3·15) in (3·14) and neglecting the terms containing η we get

$$\frac{df'^2}{d\zeta} - \frac{2\lambda}{\lambda+2\mu} \frac{f'^2}{\zeta} = \frac{4(\lambda+\mu)A^2}{(\lambda+2\mu)} \frac{1}{\zeta} \tag{3·16}$$

the other equation being satisfied identically.

Solving (3·16) we get

$$f'^2 = B\zeta^{2-2c} - \frac{2-c}{1-c} A^2 \tag{3·17}$$

where B is an arbitrary constant, and $c = 2\mu/(\lambda+2\mu)$. (3·18)

Substituting (3·15) and (3·17) in (3·7) to (3·9) we get

$$\sigma_{11} = \frac{\mu}{c}\left[3 - 2c + \frac{(3-2c)c}{1-c}\frac{A^2}{\zeta^2} - \frac{B}{\zeta^{2c}}\right] \tag{3·19}$$

$$\sigma_{22} = \sigma_{33} = \frac{\mu}{c}\left[3-2c-(1-c)\frac{B}{\zeta^{2c}}\right] \tag{3·20}$$

If the plate is bent by applying forces to the edge only, we should have

$$\sigma_{11} = o \text{ when } \zeta = \zeta_i \quad i = 1, 2. \tag{3·21}$$

which on substitution from (3·19) reduce to

$$-\frac{(3-2c)\,c}{1-c}\,\frac{A^2}{\zeta^2_i} + \frac{B}{\zeta_i^{2c}} = 3-2c,\ i = 1,\ 2 \tag{3·22}$$

Solving (3·22) we get

$$A^2 = \frac{(1-c)\,(\zeta_2^{2c}-\zeta_1^{2c})\,(\zeta_1\zeta_2)^{2-2c}}{c\,(\zeta_2^{2-2c}-\zeta_1^{2-2c})}$$

$$B = (3-2c)\,(\zeta_2^2-\zeta_1^2)\,/\,(\zeta_2^{2-2c}-\zeta_1^{2-2c})$$

The distributions of the tractions on the edge between φ and $\varphi+d\varphi$ give rise to a force P and a couple of moment M given by

$$P = \alpha\int_{\zeta_1}^{\zeta_2}\sigma_{22}\,\zeta^2\,d\zeta,\ 2M = \alpha\int_{\zeta_1}^{\zeta_2}\sigma_{22}\,\zeta^4\,d\zeta \tag{3·23}$$

which on substitution from (3·29) reduce to

$$P = \alpha\frac{\mu}{c}\left[(3-2c)\frac{\zeta_2^3-\zeta_1^3}{3} - (1-c)\,B\,\frac{\zeta_2^{3-2c}-\zeta_1^{3-2c}}{3-2c}\right] \tag{3·24}$$

$$2M = \alpha\frac{\mu}{c}\left[(3-2c)\frac{\zeta_2^5-\zeta_1^5}{5} - (1-c)\,B\,\frac{\zeta_2^{5-2c}-\zeta_1^{5-2c}}{5-2c}\right] \tag{3·25}$$

Thus we require a force P and acouple of moment M to keep the plate bent into a paraboloidal shell whose inner and outer boundaries are free from tractions.

REFERENCES

1. Lakshminarayana, G., *Proc. Nat. Acad. Sci. (India)* 29, 297-306 (1960).
2. Lakshminarayana, G. and B. Kesava Rao., *Proc. Nat. Aead. Sci. (India)* 29 (1960).
3. B. Kesava Rao, K. S. Sarma and G. Lakshminarayana., *Proc. Nat. Acad. Sci. (India) 33,* 455-460 (1963).
4. Lakshminarayana, G., *Z. A. M. M., 43,* 563-64 (1963).
5. Seth, B. R., *Phil Trans. Roy. Soc. (Kona) 234A,* 261-64 (1935).
6. Green, A. E. and W. Zerna , Theoretical Elasticity (1954).

PHYSICO-CHEMICAL PROPERTIES OF MILK : PART III COAGULATION OF MILK WITH MIXTURES OF ELECTROLYTES

By

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ABSTRACT

Coagulation of milk with different pairs of electrolytes was studied. It was found that milk shows the phenomenon of ionic antagonism when coagulated by mixtures of electrolytes. Both positive as well as negative ionic antagonisms were met with. The cations which are the coagulating ions for the negatively charged milk sol plays the important role in producing ionic antagonism.

INTRODUCTION

It has been observed by different workers (1) that coagulating powers of ions in mixtures of electrolytes on colloidal solutions are not always additive. The total amount of coagulating ions in the case of electrolytic mixture may be greater or less than the algebric sum of the individual ion taken singly. The phenomenon is known as ionic antagonism. In a previous publication (2) the coagulation of milk with different electrolytes was described. It would be interesting to find out if milk also shows ionic antagonism.

EXPERIMENTAL

Milk was collected in the same way as described in the previous paper (*loc. cit.*) and the experiments were carried out with 50% milk diluted with distilled water. First the coagulation concentrations were determined with different electrolytes at 20°C. Thereafter in each one of any set of test-tubes was placed a certain amount of the solution of one electrolyte, the amount representing a particular percentage of the coagulation concentration of that electrolyte. To this was added the other electrolyte in an increasing order of concentration. The total volume was then made up to 5 ml. with distilled water in each test-tube. This set of test-tubes was placed together with another set containing 5 ml. of 50% milk in each test-tube in an air thermostat set at 20°C. When this temperature had been attained by the test-tubes the coagulation concentration of the second electrolyte was determined as described in a previous paper (*loc. cit.*). This gave the coagulation concentration of the second electrolyte in the presence of the particular percentage of the first electrolyte.

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Experiments were performed with the foll›wing mixtures:

1. Cu $(NO_3)_2$ + HNO_3
2. Cu $(NO_3)_2$ + H_2SO_4
3. Cu $(NO_3)_2$ Al $(NO_3)_3$
4. Al $(NO_3)_3$ + HNO_3
5. Al $(NO_3)_3$ + H_2SO_4
6. Zn SO_4 + Al $(NO_3)_3$
7. Zn $(NO_3)_2$ + Al $(NO_3)_3$
8. Zn SO_4 + H_2SO_4
9. Zn SO_4 + HNO_3
10. Zn $(NO_3)_2$ + HNO_3
11. Zn $(NO_3)_2$ + Fe $(NO_3)_3$.

The average composition of the milk used for these studies was:

Fat contents	5·3%
Total solids	14·1%
Non fatty solids	8·8%

RESULTS AND DISCUSSION

TABLE I

Coagulation with Cu $(NO_3)_2$ + HNO_3 mixture

Vol. of 50% diluted milk	5 ml.	Time 1 hour
Total vol.	10 ml.	Temp. 20°C

Coag. conc. of copper nitrate 5·5 m. M./1.

Coag. conc. of nitric acid 11·6 m. M./1.

Cu $(NO_3)_2$ as %age of coag. conc.	Conc. of HNO_3 in m. M./1. Obs.	Calc.	Difference	Percentage antagonism
14·5	10·40	9·92	0·48	4·84
29·0	9·40	8·24	1·16	14·08
43·5	8·75	6·55	2·20	33·58
58·0	7·50	4·87	2·63	54·00
72·5	4·10	3·19	0·91	28·53
87·0	1·65	1·51	0·14	9·27

TABLE II

Coagulation with Cu $(NO_3)_2$ + H_2SO_4 mixture

Vol. of 50 diluted milk 5 ml. Time 1 hour

Total vol. 10 ml. Temp. 20°C.

Coag. conc. of copper nitrate 5·5 m. M./l.

Coag. conc. of sulphuric acid 6·0 m. M./l.

Cu $(NO_3)_2$ as % age of coag. conc.	Conc of H_2SO_4 in m. M./l.			Percentage antagonism
	Obs.	Calc.	Difference	
14·5	5·65	5·13	0·52	10·14
29·0	5·05	4·26	0·79	18·54
43·5	4·65	3·39	1·26	37·17
58·0	4·05	2·52	1·53	60·71
72·5	2·40	1·65	0·75	45·45
87·0	0·85	0·78	0·07	8·97

TABLE III

Coagulation of milk with $Cu(NO_3)_2$ + $Al(NO_3)_3$ mixture

Vol. of 50 % diluted milk 5 ml. Time 1 hour

Total vol. 10 ml. Temp. 20°C

Coag. conc. of copper nitrate 5·5 m. M./l.

Coag. conc. of aluminium nitrate 4·6 m. M./l.

Cu $(NO_3)_2$ as %age of coag. conc.	Conc. of Al $(NO_3)_3$ in m. M./l.			Percentage antagonism
	Obs.	Calc.	Difference	
14·5	4·35	3·93	0·42	10·70
29·0	3·95	3·27	0·68	20·80
43·5	3·45	2·60	0·85	32·70
58·0	2·35	1·93	0·42	21·75
72·5	1·34	1·26	0·08	6·35
87·0	0·62	0 60	0·02	3·33

TABLE IV

Coagulation of milk with Al $(NO_3)_3$ + HNO_3 mixture

Vol. of 50% diluted milk 5 ml. Time 1 hour

Total vol. 10 ml. Temp. 20°C.

Coag. conc. of aluminium nitrate 4·6 m. M./l.

Coag. conc. of nitric acid 11·6 m. M./l.

Al $(NO_3)_3$ as % age of coag. conc.	Conc. of HNO_3 in m. M./l.			Percentage antagonism
	Obs.	Calc.	Difference	
15·22	10·15	9·83	0·32	3·25
30·44	9·05	8·07	0·98	12·14
45·66	8·20	6·30	1·90	30·16
60·88	6·00	4·54	1·46	32·16
76·10	3·20	2·77	0·43	15·52
89·13	1·35	1·26	0·09	7·14

TABLE V

Coagulation with Al $(NO_3)_3$ + H_2SO_4 mixture

Vol. of 50% diluted milk 5 ml. Time 1 hour

Total vol. 10 ml. Temp. 20°C.

Coag. conc. of aluminium nitrate 4·6 m. M./l.

Coag. conc. of sulphuric acid 4·6 m. M./l.

Al $(NO_3)_3$ as %age of coag. conc.	Conc. of H_2SO_4 in m. M./l.			Percentage antagonism
	Obs.	Calc.	Difference	
15·22	5·30	5·08	0·22	4·33
30·44	4·65	4·17	0·48	11·51
45·66	4·20	3·26	0·94	28·83
60·88	3·30	2·35	0·95	40·42
76·10	1·80	1·43	0·37	25·87
89·13	0·70	0·65	0·05	7·69

TABLE VI

Coagulation with $ZnSO_4$ + Al $(NO_3)_3$ mixture

Vol. of 50% diluted milk 5 ml. Time 1 hour

Total vol. 10 ml. Temp. 20°C.

Coag. conc. of zinc sulphate 6·4 m. M./l.

Coag. conc. of aluminium nitrate 4·6 m. M./l.

$ZnSO_4$ as %age of coag. conc.	Conc. of Al $(NO_3)_3$ in m. M./l.			Percentage antagonism
	Obs.	Calc.	Difference	
15·00	3·65	3·91	− 0·26	− 6·65
30·00	2·70	3·22	− 0·52	− 16·15
45·00	1·90	2·53	− 0·63	− 24·90
60·00	1·30	1·84	− 0·54	− 29·34
75·00	0·62	1·15	− 0·53	− 46·08
90·00	0·12	0·46	− 0·34	− 73·91

TABLE VII

Coagulation with Zn $(NO_3)_2$ + Al $(NO_3)_3$ mixture

Vol of 50% diluted milk 5 ml. Time 1 hour

Total vol. 10 ml. Temp. 20°C.

Coag. conc. of zinc nitrate 6·5 m.M./l.

Coag. conc. of Aluminium nitrate 4·6 m. M./l.

Zn $(NO_3)_2$ as %age of coag. conc.	Conc. of Al $(NO_3)_3$ in m. M./l.			Percentage antagonism
	Obs.	Calc.	Difference	
15·38	3·65	3·89	− 0·24	− 6·17
30·77	2·70	3·19	− 0·49	− 15·36
46·15	1·85	2·48	− 0·63	− 25·40
61·54	1·25	1·77	− 0·52	− 29·38
76·92	0·60	1·06	− 0·46	− 43·40
92·36	0·10	0·35	− 0·25	− 71·43

TABLE VIII

Coagulation with $ZnSO_4 + H_2SO_4$ mixture

Vol of 50% diluted milk	5 ml.	Time 1 hour.
Total vol.	10 ml.	Temp. 20°C.

Coag. conc. of zinc. of sulphate 6·4 m. M./l.

Coag. conc. of sulphuric acid 6·0 m. M./l.

Zn SO_4 as %age of coag. conc.	Conc. of H_2SO_4 in m. M /l.			Percentage antagonism
	Obs.	Calc.	Difference	
15·00	4·90	5·10	— 0·20	— 3·92
30·00	3·75	4·20	— 0·45	— 10·71
45·00	2·65	3·30	— 0·65	— 19·70
60·00	1·70	2·40	— 0·70	— 29·17
75·00	0·85	1·50	— 0·65	— 43·33
90·00	0·20	0·60	— 0·40	— 66·66

TABLE IX

Coagulation with Zn SO_4 + HNO_3 mixture

Vol. of 50% diluted milk	5 ml.	Time 1 hour
Total vol.	10 ml.	Temp. 20°C

Coag. conc. of Zinc sulphate 6·4 m. M./l.

Coag. conc. of Nitric acid 11·6 m. M./l.

$ZnSO_4$ as %age of coag. conc.	Conc. of HNO_3 in m. M./l.			Percentage antagonism
	Obs.	Calc.	Difference	
15·00	9·30	9·86	— 0·56	— 5·68
30·00	7·20	8·12	— 0·92	— 11·33
45·00	5·10	6·38	— 1·28	— 20·06
60·00	3·45	4·64	— 1·19	— 25·65
75·00	1·75	2·90	— 1·15	— 39·65
90·00	0·50	1·16	— 0·66	— 56·85

TABLE X

Coagulation with $Zn(NO_3)_2 + HNO_3$ mixture

Vol. of 50% diluted milk 5 ml. Time 1 hour

Total vol. 10 ml. Temp. 20°C.

Coag. conc. of zinc nitrate 6·5 m. M./l.

coag. conc. of nitric acid 11·6 m. M./l.

$Zn(NO_3)_2$ as %age of coag. conc.	Conc. of HNO_3 in m. M./l.			Percentage antagonism
	Obs.	Calc.	Difference	
15·38	9·10	9·82	− 0·72	− 7·35
30·77	7·00	8·03	− 1·03	− 12·83
46·15	4·60	6·25	− 1·65	− 26·40
61·54	2·70	4·46	− 1·76	− 39·46
76·92	1·45	2·67	− 1·22	− 45·69
92·36	0·35	0·89	− 0·54	− 60·67

TABLE XI

Coagulation with $Zn(NO_3)_2 + Fe(NO_3)_3$ mixture

Vol. of 50% diluted milk 5 ml. Time 1 hour

Total vol. 10 ml. Temp. 20°C.

Coag. conc. of zinc nitrate 6·5 m. M./l.

Coag. conc. of ferric nitrate 4·8 m. M./l.

$Zn(NO_3)_2$ as %age of coag. conc.	Conc. of $Fe(NO_3)_3$ in m. M./l.			Percentage antagonism
	Obs.	Calc.	Difference	
15·38	4·25	4·06	0·19	4·68
30·77	3·65	3·32	0·33	9·94
46·15	3·00	2·58	0·42	16·28
61·54	2·35	1·85	0·50	27·03
76·92	1·20	1·11	0·09	8·11
92·86	0·38	0·37	0·01	2·70

These results show that in the case of the mixtures $Cu(NO_3)_2 + HNO_3$; $Cu(NO_3)_2 + H_2SO_4$; $Cu(NO_3)_2 + Al(NO_3)_3$; $Al(NO_3)_3 + HNO_3$; $Al(NO_3)_3 + H_2SO_4$ and $Zn(NO_3)_2$ $Fe(NO_3)_3$ positive ionic antagonism is met with, i. e. the concentrations of the second electrolyte required to coagulate milk in the presence of the first electrolyte are always higher than the calculated values. Whereas in the case of mixtures $ZnSO_4 + Al(NO_3)_3$; $Zn(NO_3)_2 + Al(NO_3)_3$; $ZnSO_4 + H_2SO_4$; $ZnSO_4 + HNO_3$ and $Zn(NO_3)_2 + HNO_3$ the reverse is the case i. e. negative ionic antagonism is observed.

Except for nitric acid the coagulating powers of individual electrolytes used in these studies are practically of the same order. All these electrolytes when taken in pairs show ionic antagonism not excluding nitric acid which has lower coagulating power. Weiser and Middleton (3) were of the opinion that the ionic antagonism is obtained when there is difference in the order of the coagulating powers of the two electrolytes but the above results do not support this view and show that ionic antagonism is not dependent on the difference in the separate coagulating powers of the two electrolytes.

Curves obtained by plotting the percentage of coagulation concentration of the first electrolyte and the percentage ionic antagonism observed with the second electrolyte are given in Fig. I, II, III and IV. In Fig. I are shown the ionic

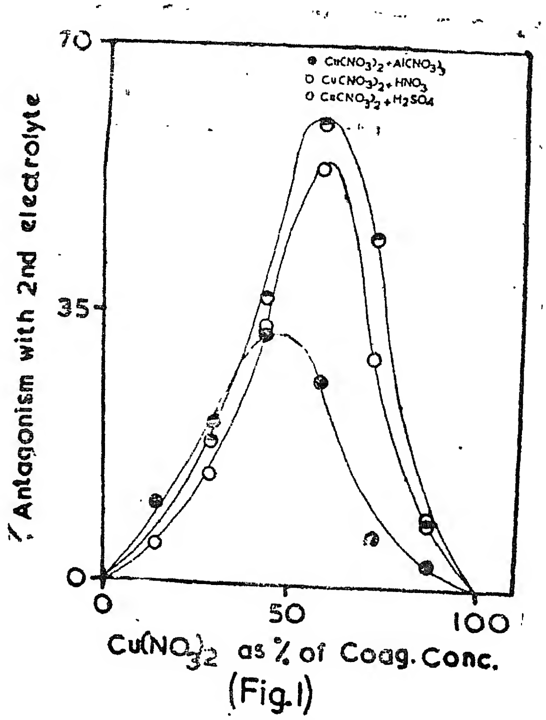


Fig. 1—Variation in the ionic antagonism observed with 2nd electrolyte in the presence of copper nitrate.

antagonisms observed with HNO_3, H_2SO_4 and $Al\,(NO_3)_3$ in the presence of $Cu\,(NO_3)_2$. It can be seen that the curves for the mixtures $Cu\,(NO_3)_2 + HNO_3$ and $Cu\,(NO_3)_2 + H_2SO_4$ are close to each other whereas the curve for $Cu\,(NO_3)_2 + Al\,(NO_3)_3$ is not. Similarly in Fig. II the curves for $Al\,(NO_3)_3 + HNO_3$ and $Al\,(NO_3)_3 + H_2SO_4$ are close to each other. In Fig. III the curves are the plots between the percentages for coagulation concentrations of $Zn\,(NO_3)_2$ and $ZnSO_4$ taken against the extent of ionic antagonism observed with $Al\,(NO_3)_3$. In this case also the two curves are quite close to each other. In Fig. IV are

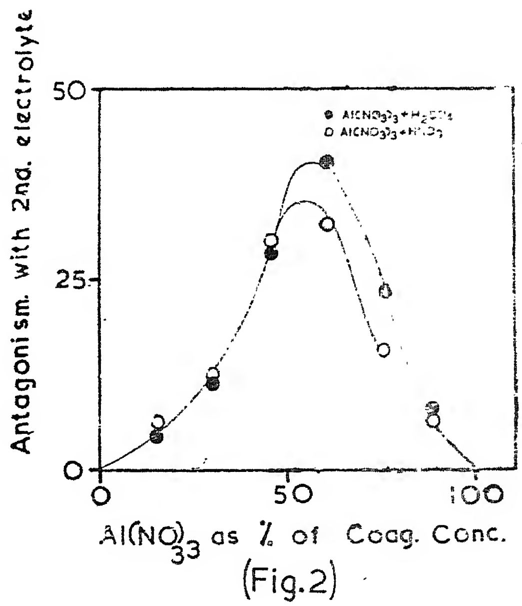
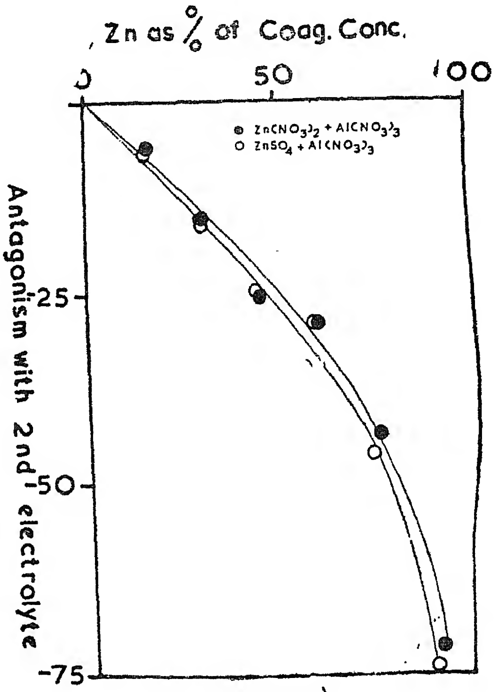


Fig. 2. Variation in the ionic antagonism observed with 2nd electrolyte in the presence of aluminium nitrate.

given the curves obtained by plotting the percentage of coagulation concentrations of $Zn\,(NO_3)_2$ and $ZnSO_4$ taken against the ionic antagonism observed with (HNO_3, H_2SO_4 and $Fe\,(NO_3)_3$. The curves for the mixtures $Zn\,(NO_3)_2 + HNO_3$; $ZnSO_4 + H_2SO_4$ and $ZnSO_4 + HNO_3$ are close to each other and show sensitization whereas for $Zn\,(NO_3)_2 + Fe\,(NO_3)_3$ it is quite different and shows stabilisation. From these curves it is observed that whenever the two coagulating cations are kept the same in the mixtures of electrolytes and the anions are changed the curves are close to each other for a particular pair of cations but when one of the cations is replaced by some other cation the curves show a change from the previous one.

On the basis of their experiments on the adsorption of cations, Ghosh and Dhar (*loc. cit.*) emphasized the role of cations in the ionic antagonism obtained with $BaCl_2$ + HCl for the antimony sulphide sol. However in their later publications they have emphasized the role of the similarly charged ions. Sen (4) also has suggested similarly charged ions to be the important factor in influencing ionic antagonism with mixtures of electrolytes. Chaudhury and Choudhuri (5) have stated that for the case of negatively charged sols the mutual cutting down in the adsorption of cations and the stabilizing effect of anions both play the important roles in the phenomenon of ionic antagonism. The results of the present investigations reveal that the cations play the important role in producing ionic antagonism with mixtures of electrolytes in the case of negatively charged milk sol.



(Fig.3)

Fig. 3. Variation in the ionic antagonism observed with 2nd electrolyte in the presence of zinc salts.

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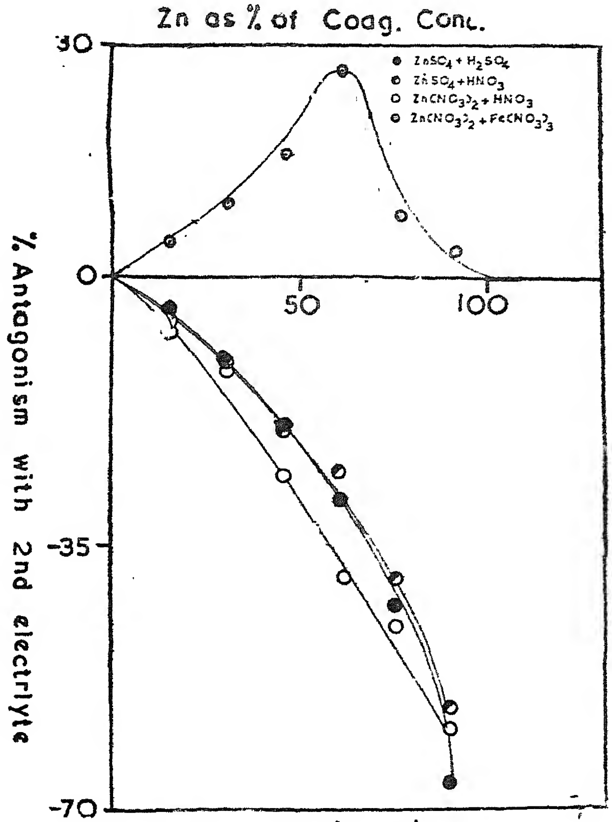

(Fig. 4)

Fig. 4. Variation in the ionic antagonism observed with 2nd electrolyte in the presence of zinc salts

REFERENCES

1. Linder and Picton, *J. Chem., Soc.*, *67*, 67 (1895).
   H. B. Weiser, *J. Phys. Chem.*, *25*, 665 (1921).
   Mukher jee and Ghosh, *J. Indian Chem., Soc.*, *1*, 213 (1925).
   N, R. Dhar and Ghosh, *J. Phys., Chem*, *29*, 435 (1925); *31*, 649 (1927).
   Pachholi and Mukherjee, *J. Indian Chem., Soc. 33*, 201 (1956).
2. Sudhish Chandra, *Proc. Natl. Acad. Sci. India*, *A 33*, 17 (1963).
3. H. B. Weiser and Midd'eton, *J. Phys. Chem.*, *24*, 68 (1920).
4. K. C. Sen, *Z. anorg. allgem Chem.*, *149*, 139 (1925); *J. Indian Chem. Soc.*, 3, 81 (1926) and *Kolloid-Z.*, *39*, 324 (1926).
5. S. C. Chaudhury and S. P. R. Choudhuri, *J. Indian Chem. Soc.*, *4*, 493 (1927).

REACTIONS OF ORTHO-ESTERS OF GERMANIUM : REACTIONS OF ETHYL ORTHOGERMANATE WITH SALICYLIC ACID

By

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ABSTRACT

Reactions between ethyl orthogermanate and salicylic acid in different stoichiometric ratios have been studied.

A physico-chemical study of the reaction between the α-hydroxy carboxylic acids and metal ions in aqueous solution carried out by Mehrotra and coworkers[1,2] revealed that the hydrogen of the hydroxyl group becomes more reactive as a result of the chelation and the complex ion dissociates as an acid. A survey of the literature reveals that during the last few years, a large amount of work has been done on the reactions of alkoxides of aluminium.[3], titanium,[4] silicon[5] and zirconium[2] with α-hydroxy carboxylic acids and salicylic acid.

In view of the above, a systematic study of the reactions of ethyl orthogermanate with salicylic acid in different stoichiometric ratios has been made by refluxing the reaction mixture under a column and collecting the ethanol-benzene azeotrope. Salicylic acid is slowly dissolved in benzene in the presence of ethyl orthogermanate in 1:1 molar ratio but on refluxing an insoluble product results which has been shown to be di-ethoxy germanium monosalicylate. When ethyl orthogermanate is caused to react with two moles of salicylic acid, only three moles of ethanol are liberated. The di-salicylate of germanium appears to withhold very firmly the remaining ethoxy group. With three moles of salicylic acid also, only three moles of ethanol were liberated and the product isolated was an equimolecular mixture of the di- and tri-salicylates. An attempt to replace the fourth ethoxy group of ethyl orthogermate was also made by taking the acid in higher proportions (more than four moles) but it was unsuccessful and the product obtained was shown to be the trisalicylate.

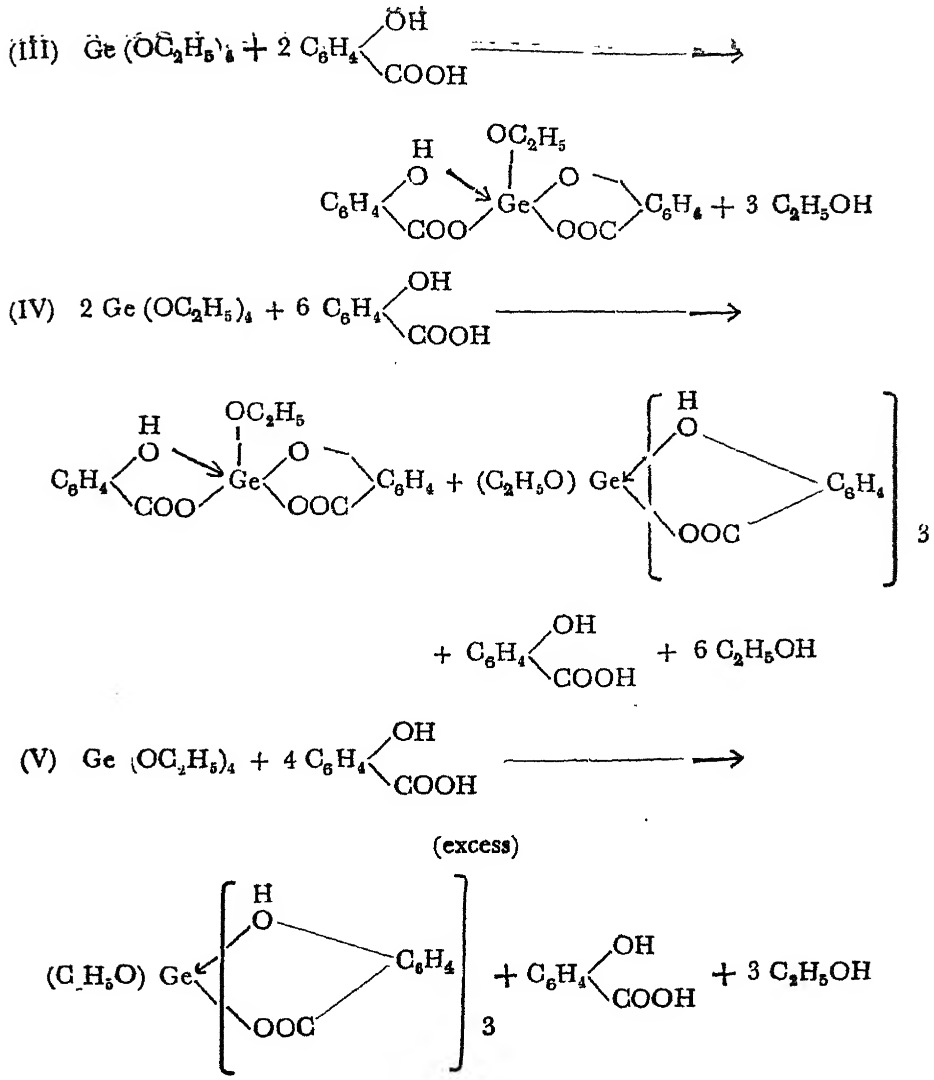
The different ethoxy salicylate derivatives of germanium, isolated during the above studies, are white solids insoluble in common organic solvents. Analytical results obtained reveal that the various reactions can be represented by the following equations :—

$$\text{(I)}\quad Ge(OC_2H_5)_4 + C_6H_4\left\langle{OH \atop COOH}\right. \xrightarrow{\text{cold}}$$

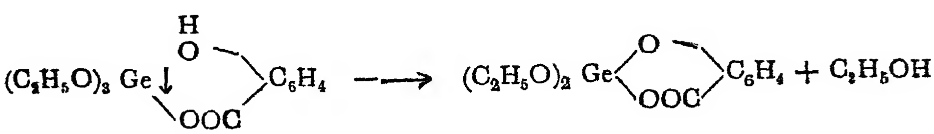
$$(C_2H_5O)_3\,Ge\left\langle{\overset{H}{O} \atop OOC}\right\rangle C_6H_4 + C_2H_5OH$$

$$\text{(II)}\quad Ge\,(OC_2H_5)_4 + C_6H_4\left\langle{OH \atop COOH}\right. \xrightarrow{\text{refluxed}}$$

$$(C_2H_5O)_2\,Ge\left\langle{O \atop OOC}\right\rangle C_6H_4 + 2\;C_2H_5OH$$



Triethoxy germanium monosalicylate on being refluxed with benzene changes into diethoxy germanium monosalicylate :—



However, the monoethoxy germanium trisalicylate remains unchanged on heating under reduced pressure (1·0 mm.) up to 50°.

EXPERIMENTAL

Apparatus :—All glass apparatus fitted with interchangeable joints was used throughout and special precautions were take to exclude moisture. Fractionations were carried out in a column packed with Raschig rings and fitted to a total condensation variable take-off still head.

Reagents:—Ethyl orthogermanate was prepared by the amonia method[6]. Bezene (B.D.H.) was dried over sodium wire and finally dried azeotropically with ethanol. Salicylic acid (A.R.) was dried at 50 – 55° at 2 mm. pressure. Ether was dried over aluminium isopropoxide and finally over sodium metal.

Analytical method :—Germanium was estimated by ignition of the compound to dioxide. Ethanol in binary azeotrope was estimated by a back titration method[7]. Salicylate was estimated by alkaline permanganate oxidation[8].

1. *Reaction between ethyl orthogermanate and salicylic acid in cold : molar ratio* 1:1.

Benzene (25 g.), ethyl orthogermanate (1·22 g.) and salicylic acid (0·67 g) were shaken vigorously for about 20 minutes to give a homogenous solution. The solvent was removed under reduced pressure and the compound was dried at 30°/1 mm. for about two hours and the residue (1·50 g) was analysed.
%, found; Ge, 21·00; Salicylate, 41·00 Calc. for $(C_2H_5O)_3.Ge\ (O_3H_5C_7)$: Ge, 21·05; salicylate, 39·75.

The above compound (1·0 g.) and benzene (30 g.) was refluxed under the column for one hour at 100° – 110°. The ethanol liberated in the reaction was withdrawn azeotropically with benzene. The insoluble residue was freed off solvent under reduced pressure and the compound was dried at 35°/2 mm. for about 1·5 hours. A white amorphous powder (0·80 g.) was obtained. Found : ethanol in the azeotrope. 0.13 g. (one mole requires 0·13 g).

%, found; Ge, 23·80; Salicylate, 45·01; Calc. for $(C_2H_5O)_2Ge(O_3H_4C_7)$; Ge, 24·30; Salicylate, 45·54.

2. *Reaction between ethyl orthogermanate aud salicylic acid : molar ratio* 1:1.

The homogeneous solution obtained by shaking benzene (40 g.), ethyl orthogermanate (1·77 g.) and salicylic acid (0·97 g.) was refluxed under the column. No sooner the bath temperature reached 80°, a white solid began to separate on the walls of reaction vessel. After refluxing the reaction mixture for about two hours at 100° – 110°, the bath temperature was raised to 120° and the azeotrope was collecvery slowly between 68° – 74°. The excess of the solvent was distilled under reduced pressure and the product was dried at 40°/5 mm. for about two hours. A white powder (2·05 g.), insoluble in benzene, ether, was obtained.

Found; ethanol in the azeotrope, 0·61 g. (2 moles require 0·64 g.)

%, found : Ge, 24·00; Salicylate, 45·40; Calc. for $(C_2H_5O)_2Ge(O_3H_4C_7)$; Ge, 24·30, salicylate, 45·54.

3. *Reaction between ethyl orthogermanate and salicylic acid : molar ratio* 1:2.

Ethyl orthogermanate (1·67 g.) was admitted to a mixture of benzene (40 g.) and salicylic acid (1·83 g.). On shaking the reaction mixture a white suspension was obtained, which was allowed to reflux under the column for about two hours at 100° — 110°.

The alcohol liberated was azeotropically removed. The excess of the solvent was distilled out. A white powder (2·5 g.) was obtained on drying the product at 28°/0·1 mm. for about two hours.

Found : ethanol in the azeotrope, 0·91 g. (3 moles require 0·91 g.)

%, found; Ge, 18·30; salicylate, 69·10; Calc. for

$$(C_2H_5O)Ge\begin{matrix} \diagup O_3H_5C_7 \\ \diagdown O_3H_4C_7 \end{matrix} \text{ : Ge, 18·57; salicylate, 69.89.}$$

4. *Reaction between ethyl orthogermanate and salicylic acid molar ratio* 1:3.

Salicylic acid (2·03 g.), ethyl orthogermanate (1·24 g.) and benzene (40 g.) were refluxed for about three hours at 110°. The azeotrope was fractionated out completely.

The reaction product was filtered and the solid was washed with dry ether and dried at 30°/5 mm. for about two hours. A white amorphous powder (2·0 g.), insoluble in benzene and ether was obtained.

Found : ethanol in the azeotrope, 0·68 (3 moles require 0·67 g.)

%, found; Ge, 15·70; salicylate, 73·71; Calc. for an equimolecular mixture of $(C_2H_5O)Ge(O_3H_5C_7)(O_3H_4C_7)$ and $(C_2H_5O)Ge(O_3H_5C_7)_3$:Ge, 15·78; salicylate, 74·42.

5. *Reaction between ethyl orthogermanate and excess (>4 moles) salicylic acid :*

Ethyl orthogermanate (1·40 g.), salicylic acid (3·21 g.) and benzene (50 g.) were caused to react at 100° — 10° for about four hours, and the ethanol produced was fractioned off as before. The insoluble white product was filtered, thoroughly washed with ether and dried at 30°/10 mm. A white powder (2·50 g., 85·3%), insoluble in benzene, was obtained.

Found; ethanol in the azeotrope, 0·78 g, (3 moles require 0·75 g.)

%, found : Ge, 14·00; salicylate, 78·70; Calc. for $(C_2H_5O)Ge(O_3H_5C_7)_3$: Ge, 13·72; salicylate, 77·75.

The above compound (1·1 g.) was heated under reduced pressure at 50°/1 mm. for about one hour; no loss in weight of the compound was observed.

%, found : Ge, 13·90.

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REFERENCES

1. Verma, I. D. and Mehrotra, R. C; *J. Ind. Chem. Soc.*, 35. 381 (1958).
2. Kapoor R. N., and Mehrotra, R. C.; *J. Amer. Chem. Soc.*, 1958, 80, 3569 ; 82, 3495 (1960).
3. Rai, A. K.; Mehrotra, R. K., and Mehrotra, R. C., *J, Prakt. Chem.*, 4, 105 (1963).
4. Verma, I. D. and Mehrotra, R. C., *J. Prakt. Chem.*, *4*, 247 (1960).
5. Mehrotra, R. C. and Pant, B. C.; (Private Communication).
6. Bradley, D. C., Kay, L., and Wardlaw, W., *J. Chem. Soc.*; 4916 (1956).
7. Bradley, D. C., Halim, F. M. A., and Wardlaw, W.; *J. Chem. Soc.*, 3450 (1950).
8. Bottger, W., Newer Methods of Volumetric Chemical Analysis, 1938 p. 61.

A THEOREM ON MEIJER'S BESSEL FUNCTION TRANSFORM

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1. In this paper two theorems about Meijer's Bessel function transform, given by Meijer [4], defined by

$$R_\nu \left[f(x) : p \right] = \int_0^\infty (p\,x)^{\frac{1}{2}}\ K_\nu\ (p\,x)\, f(x)\, dx = \bar{F}_\nu\ (p), \qquad \dots (1)$$

are proved. While illustrating the first theorem an infinite integral involving the product of G-function and Gauss's hypergeometric function is evaluated, which is the generalisation of Meijer's result [5]. Further some of its interesting particular cases are also given.

2. **Theorem 1.** If $x^{\frac{1}{2} \pm \mu} f(x) \varepsilon\, L\,(o, R)$, $R\,(\lambda) > o$,

$R\,(p) > R\,(\alpha)$, $x^{\frac{1}{2} - \lambda \pm \nu} f\,(x) \varepsilon\, L\,(o, R)$ and $f\,(x) = O\,(e^{\alpha x} x^{\rho})$ for large x, then

$$R_\nu \left[t^{-\lambda} f(t) : p \right] = \int_1^\infty \theta\,(x, p)\, \bar{F}_\mu\,(p\,x)\, dx, \qquad \dots (2)$$

where

$$\Gamma(\lambda)\, \theta\,(x, p) = 2^{1-\lambda}\, p^{\lambda}\, x^{\frac{1}{2} - \mu} (x^2 - 1)^{\lambda - 1}\, {}_2F_1 \left[\begin{matrix} \frac{1}{2}\lambda - \frac{1}{2}\mu \pm \frac{1}{2}\nu \\ \lambda \end{matrix} ; 1 - x^2 \right]$$

Proof : If Z is replaced by $p\,t$ in the result, given by Meijer [5, p. 208].

$$\int_1^\infty K_{\alpha - \beta}(Z\,x)\, {}_2F_1 \left(\begin{matrix} \alpha \pm \frac{1}{2}\nu \\ \alpha + \beta \end{matrix} ; 1 - x^2 \right) (x^2 - 1)^{\alpha + \beta - 1} x^{\alpha - \beta + 1}\, dx$$

$$= \Gamma(\alpha + \beta)\ 2^{\alpha + \beta - 1}\, Z^{-\alpha - \beta} K_\nu\ (Z),\ [R\,(\alpha + \beta) > o, R\,(p) > o] \dots (3)$$

and the value of $K_\nu\ (pt)$, so obtained, is substituted for it in the integral representation of $R_\nu\ [\ t^{-\lambda} f\,(t) : p\ ]$, then we get

$$R_\nu\ \left[t^{-\lambda}\ f(t) : p \right] = \frac{2^{1 - \alpha - \beta}\, p^{\alpha + \beta}}{\Gamma\,(\alpha + \beta)} \times$$

$$\times \int_0^\infty (p\,t)^{\frac{1}{2}} t^{\alpha+\beta-\lambda} f(t) \left\{ \int_1^\infty K_{\alpha-\beta}(p\,x\,t)\, {}_2F_1\left({\alpha \pm \frac{1}{2}\nu \atop \alpha+\beta} ; 1-x^2 \right) (x^2-1)^{\alpha+\beta-1} x^{\alpha-\beta+1}\, dx \right\} dt$$

Hence on replacing $\alpha+\beta$, $\alpha-\beta$ by λ, μ respectively, then changing the order of integration and interpreting the inner integral with the help of (1), we get the result (2).

Since (3) is absolutely convergent for $R(\alpha+\beta) > o$, $R(p) > o$ and

$$K_\nu(Z) \sim \left(\pi/(2Z) \right)^{\frac{1}{2}} e^{-Z} \text{ for large } Z$$

$$\sim A\,Z^{\nu} + B\,Z^{\nu} \text{ for small } Z,$$

hence x-integral, t-integral, as well as integral on l. h. s. are all absolutely convergent for conditions stated with the theorem and therefore the change of order of integration is admissible by de la Vallee Poussin's theorem [1, p. 504].

Corollary : If we put $\mu = \frac{1}{2}$, then since

$$R_{\pm\frac{1}{2}}\left[f(t) : p \right] = (\pi/2)^{\frac{1}{2}} L\left[f(t) : p \right] \qquad \dots (4)$$

and [2, *p.* 127].

$${}_2F_1\left({\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu,\ -\frac{1}{2}\nu-\frac{1}{2}\mu \atop 1-\mu} ; 1-Z^2 \right) = 2^{-\mu} (Z^2-1)^{\frac{1}{2}\mu}\, \Gamma(1-\mu)\, P_\nu^{\mu}(Z), \qquad \dots (5)$$

the theorems reeuces to the one given by the author [8].

Example. By taking particular values of the parameters in the result given by R. K. Saxena [10], it can be easily derived that

$$\int_0^\infty x^{2\sigma-1} K_\nu(px)\, G_{\gamma,\delta}^{\alpha,\beta}\left(z x^{2n/s} \,\middle|\, {a_1,\dots,a_\gamma \atop b_1,\dots,b_\delta} \right) dx$$

$$= \tfrac{1}{2}(2\pi)^{1-n+(1-s)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)} (2n)^{2\sigma-1} s^{\Sigma b_j - \Sigma a_j + \frac{1}{2}(\gamma-\delta)} p^{-2\sigma}$$

$$\times G_{s\gamma+2n,\, s\delta}^{s\alpha,\, s\beta+2n}\left(\frac{(2n)^{2n} s^{s(\gamma-\delta)}}{p^{2n}} z^s \,\middle|\, {\Delta(n, 1\pm\nu-\sigma),\ \Delta(s,a_1),\dots,\Delta(s,a_\gamma) \atop \Delta(s,b_1),\dots,\Delta(s,b_\delta)} \right)^* \qquad \dots (6)$$

where n and s are positive integers and $R(p) > o$, $2(\alpha+\beta) > \gamma+\delta$, $|\arg z| < (\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)\pi$, $R(\sigma \pm \nu + \frac{n}{s} b_h) > o$, $[h = 1, \dots, \alpha]$, and hence if we

start with

$$f(x) = x^{\rho-\frac{1}{2}} \; G^{k\, l}_{r, s+2n} \left(\frac{Zx^{2n}}{(2n)^{2n}} \,\middle|\, \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s, \Delta\left(n, \frac{1-\rho\pm\mu}{2}\right) \end{matrix} \right)$$

then

$$\overline{F}_{\mu}(p) = \tfrac{1}{2}(2\pi)^{1-n}(2n)^{\rho}\, p^{-\rho\;\frac{1}{2}}\, G^{k,l}_{r,s}\left(zp^{-2n} \,\middle|\, \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right)$$

and

$$R_{\nu}\left[t^{-\lambda} f(t) : p \right] = \tfrac{1}{2}(2\pi)^{1-n}(2n)^{\rho-\lambda}\, p^{\lambda\;\;\rho-\frac{1}{2}}$$

$$\times\; G^{k,\, l\,+\,2n}_{r+2n,\, s+2n} \left(\frac{Z}{p^{2n}} \,\middle|\, \begin{matrix} \Delta\left(n, \frac{1-\lambda-\rho\pm\nu)}{2}\right) a_1, \ldots, a_r \\ b_1, \ldots, b_s, \Delta\left(n, \frac{1-\rho\pm\mu)}{2}\right) \end{matrix} \right),$$

so that result (2) gives us, on making little adjustment of the parameters

$$\int_1^{\infty} v^{d-1}(v-1)^{a+b-c-d-1}\; {}_2F_1\left(\begin{matrix} a-c,\, b-c \\ a+b-c-d \end{matrix} ; 1-v \right) G^{k,l}_{r,s}\left[Zv^{-n} \,\middle|\, \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right] dv$$

$$= \Gamma(a+b-c-d)\, n^{c+d-a-b}\, G^{k,l+2n}_{r+2n,\, s+2n}\left(Z \,\middle|\, \begin{matrix} \Delta(n,a), \Delta(n,b), a_1, \ldots, a_r \\ b_1, \ldots, b_s, \Delta(n,c), \Delta(n,d) \end{matrix} \right), \quad (7)$$

* $\Delta(s,a)$ stands for the sequence of parameters $\frac{a_1}{s}, \ldots, \frac{a_1+s-1}{s}$,

and $\Delta(s, a\pm b) \equiv \Delta(s, a+b), \Delta(s, a-b)$

where n is a positive integer, $2(k+l) > r+s$, $R(a+b-c-d) > 0$.

$R(nb_h - a + 1) > 0$, $R(nb_h - b + 1) > 0$ $[h = 1, \ldots, k]$, $|\arg z| < (k + l - \frac{1}{2}s - \frac{1}{2}r)\pi$

Some of the interesting particular cases of the above are given below :

(*a*) If we put $n = 1$ and use the result [2, p. 209]

$$G^{m,n}_{p,q}\left(Z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, q_q \end{matrix} \right) = G^{n,m}_{q,p}\left(\frac{1}{Z} \,\middle|\, \begin{matrix} 1-b_1, \ldots, 1-b_q \\ 1-a_1, \ldots, 1-a_p \end{matrix} \right), \quad \ldots (8)$$

we get the result given by Meijer [5, *p*. 199].

(*b*) Since [2, p. 215]

$$G^{p,1}_{q+1,p}\left(x \,\middle|\, \begin{matrix} 1, \beta_1, \ldots, \beta_q \\ \alpha_1, \ldots, \alpha_p \end{matrix} \right) = E(p; \alpha_r : q; \beta_s : x) \quad \ldots (9)$$

when $k = 1$, $l = p = r$, $s = q + 1$ then by virtue of (8) and (9) the (7) reduces to the result given by C. B. Rathie [7], while if we take $l = 1$, $k = s = p$, $r = q + 1$ and make suitable substitution for parameters, we get

$$\int_1^{\infty} v^{\rho-1} \; (v-1)^{c-1} \; {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; 1-v \right) E\,(p\,;\, \alpha_r : q\,;\, \beta_s : Zv^{-n})\, dv$$

$$= \Gamma\,(c) n^c \; G^{p,\,2n+1}_{2n+1+q,\,2n+p}\left(Z \middle| \begin{matrix} \Delta\,(n,\, c-b+\rho),\, \Delta(n,\, c-a+\rho),\, 1,\, \beta_1,\, \ldots,\, \beta_q \\ \alpha_1,\, \ldots,\, \alpha_p,\, \Delta\,(n,\, c-a-b+\rho),\, \Delta\,(n,\, \rho) \end{matrix} \right) \quad \ldots(10)$$

where n is a positive integer, $R\,(c) > 0$, $p + 1 > q$, $|\arg z| < (p + 1 - q)\frac{\pi}{2}$,

$R\,(n\alpha_r + b - c - \rho + 1) > 0$, $R\,(n\alpha_r - c + a - \rho + 1) > 0$.

The right hand side of (10) can be expressed as the sum of $(2n + 1)$ terms of E-functions, if $q - 1 < p \leqslant q + 1$ and the sum of p E-functions if $p > q + 1$.

(c) If we take $k = l = r = s = 2$, $a_1 = 1 - \alpha$, $a_2 = 1 = \beta$, $b_1 = 0$ and $b_2 = \gamma - \alpha - \beta$ in (7), then by virtue of the identity [9].

$$G^{2,\,2}_{2,\,2}\left(Z \middle| \begin{matrix} 1 - a,\, 1 - b \\ 0,\, c - a - b \end{matrix} \right) = \frac{\Gamma\,(a)\;\Gamma\,(b)\;\Gamma\,(c - a)\;\Gamma\,(c - b)}{\Gamma\,(c)} \; {}_2F_1\,(a,b;c;1-z), \quad \ldots(11)$$

we have

$$\int_1^{\infty} v^{d-1} \; (v - 1)^{a+b-c-d-1} \; {}_2F_1 \; \left(\begin{matrix} a-c,\, b-c \\ a+b-c-d \end{matrix} ; 1-v \right) \; {}_2F_1 \left(\begin{matrix} \alpha,\, \beta \\ \gamma \end{matrix} ; 1 - \frac{Z}{v^n} \right) dv$$

$$= \frac{\Gamma\,(a + b - c - d)\;\Gamma\,(\gamma)}{\Gamma\,(\alpha)\;\Gamma\,(\beta)\;\Gamma\,(\gamma - \alpha)\;\Gamma\,(\gamma - \beta)} \; n^{c+d-a-b}$$

$$G^{2,\,2n+2}_{2n+2,\,2n+2}\left(Z \middle| \begin{matrix} \Delta\,(n,\, a),\, \Delta\,(n,\, b),\, 1-\alpha,\, 1-\beta \\ 0,\, \gamma-\alpha-\beta,\, \Delta\,(n,\, c),\, \Delta\,(n,\, d) \end{matrix} \right) \quad \ldots(12)$$

where n is a positive integer, $R\,(a+b-c-d) > 0$, $R(a) < 1$, $R\,(b) < 1$, $|\arg z| < \pi$.

(d) Further due to (5), when $a = \frac{3}{4} + \frac{1}{2}\sigma + \frac{1}{2}\nu - \frac{1}{2}\mu$,

$b = \frac{3}{4} + \frac{1}{2}\sigma - \frac{1}{2}\nu - \frac{1}{2}\mu$, $c = \frac{1}{2} + \frac{1}{2}\sigma$. $d = \frac{1}{2}\sigma$, $\gamma = 1 - \rho$, $\alpha = \frac{1}{4} + \frac{1}{2}\lambda - \frac{1}{2}\rho$,

$\beta = \frac{1}{4} - \frac{1}{2}\lambda - \frac{1}{2}\rho$, $v = u^2$, $z = y^2$

(12) yields the result

$$\int_1^{\infty} u^{\sigma-n\rho-1} (u^2-1)^{-\frac{1}{2}\mu} (y^2-u^{2n})^{\frac{1}{2}\rho} \; P^{u}_{v-\frac{1}{2}}(u)\; P^{\rho}_{\lambda-\frac{1}{2}}(y/u^n)\, du$$

$$\frac{2^{\mu-\rho-2}\; n^{\mu-1}}{\pi\;\Gamma(\frac{1}{2}\pm\lambda-\rho)}\; G^{2,\,2n+2}_{2n+2,\,2n+2}\left(\begin{matrix}\Delta\,(n,\frac{3}{4}\pm\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}\sigma),\ \frac{3}{4}\pm\frac{1}{2}\lambda+\frac{1}{2}\rho\\ 0,\ \frac{1}{2},\ \Delta\,(2n,\sigma)\end{matrix}\right), \quad \dots \quad (13)$$

where n is a positive integer, $R\,(1-\mu)>0$, $R\,(\frac{1}{2}+\mu-\sigma\pm\nu)>0$.

(*e*) While if we take $a=\frac{3}{4}+\frac{1}{2}\sigma+\frac{1}{2}\nu-\frac{1}{2}\mu$, $b=\frac{3}{4}+\frac{1}{2}\sigma-\frac{1}{2}\nu-\frac{1}{2}\mu$ $c=\frac{1}{2}+\frac{1}{2}\sigma$, $d=\frac{1}{2}\sigma$, $\alpha=-\frac{1}{2}\lambda-\frac{1}{2}\rho$, $\beta=\frac{1}{2}-\frac{1}{2}\lambda-\frac{1}{2}\rho$, $\gamma=1-\rho$ substitute $v=u^2$ replace z by y^{-2} in (12), then due to (5) and the result [2, p. 129].

$${}_2F_1(-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu, \tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu\,;\,1-\mu\,;\,1-z^{-2})$$

$$=2^{-\mu}\,(z^2-1)^{\frac{1}{2}\mu}\,z^{-\nu-\mu}\,\Gamma(1-\mu)\,P^{\mu}_{\nu}(z), \quad \dots \quad (14)$$

we get

$$\int_1^{\infty} u^{\sigma-n\lambda-n\rho-1}(u^2-1)^{-\frac{1}{2}\mu}(y^2u^{2n}-1)^{\frac{1}{2}\rho}\;(P^{\mu}_{v-\frac{1}{2}}(u)\;P^{\rho}_{\lambda}(yu^n)\,du$$

$$=\frac{2^{\mu-\rho-2}\,y^{\lambda+\rho}\,n^{\mu-1}}{\pi\;\Gamma(-\lambda-\rho)\,\Gamma(1+\lambda-\rho)}\times$$

$$+\;G^{2,\,2n+2}_{2n+2,\,2n+2}\left(y^{-2}\left|\begin{matrix}\Delta\,(n,\frac{3}{4}+\frac{1}{2}\sigma\pm\frac{1}{2}\nu-\frac{1}{2}\mu),\ \Delta\,(2,\,1+\lambda+\rho)\\ 0,\ \frac{1}{2}+\lambda,\ \Delta\,(2n,\sigma)\end{matrix}\right.\right), \quad \dots \quad (15)$$

where n is a positive integer, $R\,(1-\mu)>0$, $R\,(\frac{1}{2}+\mu\pm\nu-\sigma)>0$.

If we put $n=1$, $\sigma=\lambda+\rho+1$, then it reduces to the result given by the author [8].

(*f*) If we put $\nu=\frac{1}{2}$, $\lambda=0$ in (15), then by virtue of the results

$$\Gamma(1-\mu)\,P^{\mu}_{0}(z)=\left(\frac{Z+1}{z-1}\right)^{\frac{1}{2}\mu}, \quad \dots \quad (16)$$

$$G^{2m,\,2n}_{2p,\,2q}\left(2^{2p-2q}\,x^2\left|\begin{matrix}\frac{1}{2}a_r,\ \frac{1}{2}+\frac{1}{2}a_r\\ \frac{1}{2}b_s,\ \frac{1}{2}+\frac{1}{2}b_s\end{matrix}\right.\right)$$

$$= (2\pi)^{m+n-\frac{1}{2}p-\frac{1}{2}q} \; \overset{\cdot}{\underset{2}{\Sigma}}\, a_r - \Sigma\, b_h + \tfrac{1}{2}(q-p) - 1 \quad G^{m,\,n}_{p,\,q}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix} \right), \quad \ldots (17)$$

$$G^{1,\,p}_{p,\,q+1}\left(x \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ o, 1-b_1, \ldots, 1-b_q \end{matrix} \right) = \frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{q} \Gamma(b_j)} \; {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} : -x \right) \quad \ldots (18)$$

and

$$\prod_{r=o}^{n-1} \Gamma(z + r/n) = (2\pi)^{\frac{1}{2}(n-1)} \; n^{\frac{1}{2}-nz} \; \Gamma(nz), \quad \ldots (19)$$

we get, on substituting $u = 1/t$, $y = 1/z$, for $R(1-\mu) > o$, $R(\mu - \sigma) > o$

$$\int_o^1 t^{\mu-\sigma-1} \; (1-t)^{-\mu} \; (1+zt^n)^{\rho} \; dt$$

$$= \frac{\Gamma(\mu-\sigma)\,\Gamma(1-\mu)}{\Gamma(1-\sigma)} \quad {}_{n+1}F_n \left(\begin{matrix} \Delta(n, \mu-\sigma), -\rho \\ \Delta(n, 1-\sigma) \end{matrix} : -z \right), \quad \ldots (20)$$

(g) If we substitue $v = 1 + t/p$, $z = yp^{-n}$ take $b = d$, $c = \rho$ and $a = \rho + \xi$ in (7), it reduces to

$$\int_o^{\infty} t^{\xi-1} \, (p+t)^{\rho-1} \quad G^{k,\,l}_{r,\,s}\left(y\,(p+t)^{-n} \,\middle|\, \begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right) dt$$

$$= \Gamma(\xi)\, u^{-\xi} \, p^{\xi+\rho-1} \; G^{k,\,l+n}_{r+n,\,s+u}\left(yp^{-n} \,\middle|\, \begin{matrix} \Delta(n, \xi+\rho), a_1, \ldots, a_r \\ b_1, \ldots, b_s, \Delta(n, \xi) \end{matrix} \right), \quad \ldots (21)$$

where $R(\xi) > o$, $R(\xi + \rho - n\, b_h) > o$ $[h = 1, \ldots, k]$ and n is a positive integer.

If we apply the Mellin inversion formula [12] to (21), we get.

$$\frac{1}{2\pi i} \int_{c-\infty}^{c+i\infty} \Gamma(\xi)\,(nt/p)^{-\xi} \quad G^{k,\,l+n}_{r+a,\,s}\left(yp^{-n} \,\middle|\, \begin{matrix} \Delta(n, \rho+\xi), a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right) d\xi$$

$$= (1 + t/p)^{\rho-1} \quad G^{k,\,l+n}_{r+n,\,s}\left[y\,(p+t)^{-n} \,\middle|\, \begin{matrix} \Delta(n, \rho), a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \right] \quad \ldots (22)$$

Theorem 2. If $x^{\frac{1}{2} \pm \mu} f(x) \varepsilon L(O, R)$, $x^{-\rho \pm \nu - 3/2} f(x) \varepsilon L(O, R)$ $p > o$, $R(1 \pm \mu \pm \nu - \rho) > o$ then

$$R_\nu \left[x^{-\rho-2} f(x) : p \right]$$

$$= 2^\rho \; p^{\frac{1}{2}} \int_0^\infty x^{-\rho-\frac{1}{2}} S_2 \left(\tfrac{1}{2}\nu - \tfrac{1}{2}, -\tfrac{1}{2}\nu - \tfrac{1}{2}, \frac{\rho+\mu}{2}, \frac{\rho-\mu}{2} \cdot \tfrac{1}{4} p x\right)$$

$$R_\mu \left[f(t) : x \right] dx \qquad \ldots (23)$$

Proof: By virtue of the result

$$2^{-\alpha-\beta} \; K_\nu(z)$$

$$= \int_0^\infty K_{\alpha-\beta}(v) \, S_2 \left(-\tfrac{1}{2}\nu - \tfrac{1}{2}, \tfrac{1}{2}\nu - \tfrac{1}{2}, \beta, \alpha : \tfrac{1}{4} z v\right) v^{-\alpha-\beta} \; dv, \; (24)$$

given by Meijer [6], we have

$$R_\nu \left[x^{-\rho-2} f(1/x) : p\right]$$

$$= \int_0^\infty (p x)^{\frac{1}{2}} K_\nu (p x) \; x^{-\rho-2} f(1/x) \, dx$$

$$= \int_0^\infty (px)^{\frac{1}{2}} \; x^{-\rho-2} f(1/x) \left\{ \int_0^\infty K_{\alpha-\beta}(t/x) \, S_2 \left(-\tfrac{1}{2} \pm \tfrac{1}{2}\nu, \beta, \alpha : x\, pt\right) t^{-\alpha-\beta} \right.$$

$$\left. (2x)^{\alpha+\beta} \, x \, dt \right\} dx$$

$$= p^{\frac{1}{2}} \, 2^{\alpha+\beta} \int_0^\infty t^{-\alpha-\beta-\frac{1}{2}} S_2 \left(-\tfrac{1}{2} \pm \tfrac{1}{2}\nu, \beta, \alpha : \tfrac{1}{4} pt\right) \left\{ \int_0^\infty (t/x)^{\frac{1}{2}} K_{\alpha-\beta}(t/x) \right.$$

$$\left. x^{\alpha+\beta-\rho-2} f(1/x) \, dx \right\} dt.$$

Now if we substitute $x = u$, $\alpha - \beta = \mu$, $\alpha + \beta = \rho$ in the inner integral and interprete it with the help of (1), we get (21).

Corollaries: If we take $\nu = \frac{1}{2}$ and $\rho = -\lambda - 1$, the theorem reduces to one given by R. K. Sexena [11]. Further if we take $\mu = \frac{1}{2}$ in addition to $\nu = \frac{1}{2}$, then we get the theorem by H. C. Gupta [23].

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REFERENCES

1. Bromwich, T J. I'A., Theory of Infinite series, London, 1947.
2. Erdelyi, A., Higher Transcendental Functions Vol. I, McGraw Hill Co. New York, 1953.
3. Gupta, H. C., On operational calculus, *Proc. Nat. Inst. Sc., India*, **14**, 1948, 131-156.
4. Mijer, C. S., Ueber eine Ewirterung der Laplace Transformation I, *Proc. kon. Neder. Akad. v. Wet.*, 43, 1940, 599-608.
5. Meijer, C. S., Ueber Besselsche, Struvesche und Lommelsche Funktionen, *Proc. kon. Neder, Akad. v. Wet.*, 43, 1940, 198-210.
6. Meijer, C. S., Integraldarstellungen for Struvesche und Besselsche Funktionen, *Compisito Mathematica* 6, 1939, 348-367.
7. Rathie, C. B., A theorem in Operational Calculue and some integrals involving Legendre, Bessel and E-function, *Proc. Glasgow Math. Assos.*, 2, 173-179, 1956.
8. Sharma, K. C., Infinite integrals involving product of Legendre functions, *Proc. Glasgow Math Assos.*, 3, 1957, 111-118.
9. Sharma, K. C., A few Inversion Formula for Meijer's Bassel Function Transform, *Proc. Nat. Inst. Sc. India*, 30, 1964, 736-742.
10. Saxena, R. K., Some theorems on Generalised Laplace Transform I, *Proc. Nat. Inst. Sc. India*, 26, 1960, 400-413.
11. Saxena, R. K., A theorem on Meijer Transform, *Proc. Nat. Inst. Sc. India*, 25, 1959, 166-170.
12. Titchmarsh, E. C., Introduction to the Theory of Fourier Integral, Oxford, London, 1948.

SOME PROPERTIES OF SOLUTIONS OF DIFFERENTIAL SYSTEMS

By

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ABSTRACT

In this paper, we shall discuss the ultimate boundedness and asymptotic stability of a differential system with respect to its approxime to system.

§ 1. This is a continuation of the author's work [3], [4] in this field. In this note some results for the stability and boundedness of solutions are derived. Our discussion depends on the following simple lemma proved in [3]. It will be stated in a different form so as to suit our purpose.

Lemma: Let the function $\omega(t, \theta(t)$ be continuous and defined for $t_0 \leqslant t < \infty$, $\theta > 0$. Suppose $M(t)$ is the maximal solution of

$$(1\cdot1) \qquad z^1 = \omega(t, \theta(t)) \qquad M(t_0) = M_0$$

Where $\theta(t) = z(t) + y(t)\, e^{\alpha t}$. where α is some const. >0.

Let $m(t)$ and $y(t)$ he vectors ; continuous functions on the same range and satisfy

$$\limsup_{h \to 0^+} \left(\frac{1}{h}\right)\left[m(t+h) - m(t)\right] \leqslant \omega\left(t, M(t) + \| y(t) \| e^{\alpha t}\right)$$

Then,

$$m(t) \leqslant M(t)$$

§ 2. Consider the differential systems

$$(2\cdot1) \qquad x^1(t) = p(x, t) + q(x, t) \qquad x(t_0) = x_0\ (t_0 \geqslant 0)$$

$$(2\cdot2) \qquad y^1(t) = p(y, t) \qquad y(t_0) = y_0$$

where x, y, p and q are n-dimentional vectors and where p and q are continuous functions on the product space $\Delta = I \times R^n$.

Let I be the interval $O \leqslant t < \infty$; and R^n be the n-dimentional Euclidean space. Also let $R^+ = [o, +\infty)$.

Let $\| x \|$ denote the norm of the element 'x'.

The following definition are required before proceeding further.

Let $x(t)$ and $y(t)$ be any two solutions of the systems (2·1) and (2·2) respectively.

(d_1) The system (2·1) or (2·2) is said to be equi – norm – bounded with respect to the system (2·2) or (2·1) if for each α greater than zero and $t_0 \geqslant 0$, there exists a positive function $\beta(t_0, \alpha)$ continuous in t_0, for each α, satisfying

$$\| x(t) - y(t) \| < \beta(t_0, \alpha)$$

for all $\| x_0 - y_0 \| \leqslant \alpha$ and $t \geqslant t_0$.

(d_2) If β in (d_1) is independent of t_o, the system (2·1) or (2.2) is said to be uniform-norm-bounded with respect to the system (1·2) or (1·1).

(d_3) The system (2·1) or (2·2) is said to be equi-stable with respect to the system (2·2 or (2·1) if for each $\epsilon > 0$ and $t_o \geqslant 0$, there exists a positive function $\eta(t_o, \epsilon)$ continuous in t_o, for each ϵ, satisfying

$$\|x(t) - y(t)\| < \epsilon$$

for all $\|x_o - y_o\| \leqslant \eta(t_o, \epsilon)$ and $t \geqslant t_o$.

(d_4) If η in (d_3) in independent of t_o, the system (2·1) or (2·2) is said to be uniform-stable with respect to the system (2·2) or (2·1)

(d_5) The system (2·1) or (2·2) is said to be quesi-equi-ultimately-norm-bounded with the system (2·2) or (2·1) if, for each $\alpha > 0$ and $t_o \geqslant 0$ there exists positive numbers B and T (t_o, α, B) such that

$$\| x(t) - y(t) \| < B$$

$$\text{for all } \| x_o - y_o \| \leqslant \alpha \text{ and } t > t_o + T(t_o, \alpha, B)$$

(d_6) If T in (d_5) is independent of t_o, the system (2·1) or (2·2) is said to be quasi-uniform-ultimately-norm-bounded with respect to the system (2 2) or (2·1)

(d_7) when $(d1)$ and $(d5)$ hold simultaneously, the system (2·1) or (2·2) is said to be equi-ultimately-norm-bounded with respect to the system (2·2) or (2·1)

(d_8) when $(d2)$ and $(d6)$ hold slmultaneously the system (2·1) or (1·2) is said to be uniform-ultimately-norm-bounded with respect to the system (2·2) or (2·1)

(d_9) The system (2·1) or (2·2) is said to be quasi-equi-asymptotically-stable with respect to the system (2·2) or (2·1) if for each $\epsilon > 0$, $\alpha > 0$ and $t_o \geqslant 0$ there exists a positive number T (t_o, ϵ, α), satisfying

$$\| x(t) - y(t) \| < \epsilon$$

$$\text{for all } \| x_o - y_o \| \leqslant \alpha \text{ and } t > t_o + T(t_o, \epsilon, \alpha).$$

(d_{10}) If T in (d_9) is independent of t_o, the system (2·1) or (2·2) is said to be quasi-uniform-asymptotically-stable with respect to the system (2·2) or (2·1).

(d_{11}) If $(d3)$ and $(d9)$ hold simultaneously, the system (2·1) or (2·2) is said to be equi-asymptotically-stable with respect to the system (2·2) or (2·1)

(d_{12}) If $(d4)$ and $(d10)$ hold simultaneously the system (2·1) or (2·2) is said to be uniform-asymptotically-stable with respect to the system (2·2) or (2·1)

Remarks: Now suppose that $p(y, t) \equiv 0$ and $y \in S$ where S is a non-empty set in R^n

Let $d(x, s)$ be the distance between the point 'x' and the set 's' defined by $d(x,s) = \inf [\| x - y \|, y \in s]$.

The definitions $(d1)$ to $(d12)$ above can be reformulated. For example $(d3)$ would run as follows

($d3$) The system (2·1) is said to be equi-stable with respect to the set $I \times S$, if for each $\epsilon > 0$ and $t_o \geqslant 0$ there exists a positive function $\eta(t_o, \epsilon)$ continuous in t_o satisfying

$$d(x(t), s) < \epsilon$$

for all $d(x_0, s) \leqslant \eta(t_o, \epsilon)$ and $t \geqslant t_0$.

Since $d(x, s) \leqslant \| x - y \|$ for all $y \in s$, the results given below would include the stability and boundedness of a set. If further $s = o$, there results precipitate into ordinary stability and boundedness of the origin. Thus our definitions, the formation of which was suggested by the work of Yoshizawa [6] and [7] and more general than those in [6] and [7].

§ 3. *Theorem* 1: Let the function $\omega\left(t, z + y e^{\alpha t}\right)$ be continuous, non-decreasing in $z + y e^{\alpha t}$ and defined on $I \times R^+$. Supose

$$(3\cdot1) \quad \begin{aligned} &(i) \; \| x - y + h\,[p(x, t) - p(y, t)] \| \leqslant \| x - y \| (1 - \alpha h) \\ &(ii) \; \| q(t, x) \| \leqslant \omega\left(t, \| x(t) \| e^{\alpha t}\right) e^{-\alpha t} \end{aligned}$$

where α is some constant >0 and h is sufficiently small positive quantity.

Let $M(t)$ be the maximal solution of

$$(3\cdot2) \qquad z' = \omega\left(t, z + y e^{\alpha t}\right) \qquad M(t_o) = M_o$$

Then, if $x(t)$ and $y(t)$ are any two solutions of (2·1) and (2·2) such that $\| x_o - y_o \| \leqslant M_o$, We have

$$\| x(t) - y(t) \| e^{\alpha t} \leqslant M(t)$$

Proof : Let $m(t) = \| x(t) - y(t) \| e^{\alpha t}$ where $x(t)$ and $y(t)$ are any two solutions of (2·1) and (2·2) such that $\| x_o - y_o \| \leqslant Mo$. Then, we have

$$m(t + h) - m(t) \leqslant e^{\alpha t} \Big[e^{\alpha h} \Big\{ \| x - y + h \big[P(x, t) - P(y, t) \big] \| + h \| q(x, t) \| + \| \epsilon_1 h \| + \| \epsilon_2 h \| \Big\} - \| x - y \| \Big]$$

where ϵ_1 and ϵ_2 tend to zero as 'h' tends to zero.

Using (3·1) (i) and (ii) it follows that

$$\lim_{h \to 0^+} \text{Sup} \left(\frac{1}{h} \right) \big[m(t + h) - m(t) \big] \leqslant \omega\left(t, \| x(t) \| e^{\alpha t}\right)$$

Since $\| x \| - \| y \| \leqslant \| x - y \|$ and by the monotonic character of ω we have

$$\lim_{h \to 0^+} \text{Sup} \left(\frac{1}{h} \right) \big[m(t + h) - m(t) \big] \leqslant \omega(t, m(t) + \| y(t) \| e^{\alpha t}$$

Then the application of Lemma, yields

$$\| x(t) - y(t) \| e^{\alpha t} \leqslant M(t)$$

Note Put $\alpha = 0$, we get thereon 2 in [4].

Theorem 2 : Suppose that the assumptions of Therem 1 hold. Suppose further that (*a*) the differential equation (3·2) is (a_1) equi-bounded (a_2) uniformly bounded, then the systems (2·1 and (2·2) satisfy the definitions (d_1) and (d_2) and hence (d_7) and (d_8).

(*b*) the identically zero solution of (3·2) is (b_1) equi-stable (b_2) uniformly-stable, then the systems (2·1) and (2·2) satisfy the definitions (d_3) and (d_4) and hence (d_{11}) and (d_{12}).

Proof : Suppose that the differential equation (3·2) is equi-bounded. Suppose $x(t)$ and $y(t)$ are any two solutions of (2·1) and (2·2) such that $\| x_o - y_o \| \leqslant M_0$. Then we have from theorem 1 that

$$\| x(t) - y(t) \| \quad e^{at} \leqslant M(t) \text{ for } t \geqslant t_o.$$

For this it is easy to see that that the system (2·1) and (2·2) satisfy the definitions (d_1), (d_2), (d_3) and (d_4). In view of the definitions (d_7), (d_8), (d_{11}) and (d_{12}), the proof would be complete, if we show that the systems also satisfy (d_5), (d_6), (d_9) (d_{10}). For this purpose, suppose that $\{t_k\}$ is a divergent sequence. Assume if possible that $\| x(t_k) - y(t_k) \| \geqslant L$

Then we get

$$L\, e^{at_k} \leqslant M(t_k) < \beta\ (t_o, \delta)$$

This leads to a contradiction as $k \rightarrow \infty$, since L and β are positive. Since L is arbitrary, this shows that the systems satisfy the definition (d_5). Similar arguments hold for the other cases, and the theorem is proved.

Remarks . If $p(y, t) \equiv O$, we get the same concepts relative to an arbitrary solutions of

$$x' = q(x, t)$$

It is interesting to note that Messera [1], [2] has shown the equivalence of some of the types of stability when the right hand side of (2·2) is linear, periodic or of the first order ($n = 1$) and also gives examples. Further, if 'p' is linear as pointed out in [5, P 10.] the boundedness and stability properties of (2·2) are equivalent. Hence the boundedness and stability properties of (2·1) depend only on (3·2) even if the system (2·2) satisfy either boundedness or stability in this case [if p is linear].

In the end, I wish to thank Prof. V. Lakshmikantham for suggesting improvements and constructive criticism.

REFERENCE

1. Messera. J. L.: On Liapounoff's conditions of stability, *Ann. of Math.* (2) 50 (1949) 705-721.

2. ———————: Contributions to stability Theory, *Ann. of Math*, (2) 64 (1956) 182-206.

3. M. Rama Mohana Rao : A Note on an integral inequality : *Jr. of Ind. Math Soc.* Vol. 27 (1963) 67-69.

4. ———————: Some problems on systems of ordinary differential equations, *Proc. Nat. Acad. Sci.* (India) Vol. 34 (1964) 229-232.

5. Yoshizawa, T: Liapunov functions and boundedness of solutions. *Funk. Ekav.* 2 (1959) 95-142.

6. ———————: Stability and Boundedness of systems : *Arch. Rat. Mech. Anal.* 6 (1960) 409-421.

7. ———————: Asymptotic stability of solutions of non-autonomous system near set. *Jr. Math. Kyoto. Univ.* 1 (1962) 303-323.

A RELATION BETWEEN WHITTAKER AND GENERALIZED STIELTJES TRANSFORMS

By

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ABSTRACT

A theorem giving relation between Whittaker and Generalized Stieltjes transforms has been proved here.

1. Verma (7, p. 17) gave the generalization of the classical Laplace transform

$$\varphi(s) = s \int_0^\infty e^{-st} f(t)\, dt \tag{1·1}$$

in the form

$$\varphi(s) = s \int_0^\infty (2st)^{-\frac{1}{2}}\, W_{k,m}(2st) f(t)\, dt; \tag{1·2}$$

which is known as Whittaker transform. With $k = \frac{1}{4}$, $m = \pm\frac{1}{4}$, (1·2) reduces to (1·1) and is represented symbolically as $\varphi(s) \underset{m}{\overset{k}{=}} f(t)$.

Generalized Stieltjes transform of order σ is defined by the equation

$$\Psi(s;\sigma) = G_\sigma \left\{ f(t); s \right\} = \int_0^\infty f(t)\,(t+s)^{-\sigma}\, dt. \tag{1·3}$$

The object of this paper is to obtain a relation between the Whittaker transform of $f(t)$ and the generalized Stieltjes transform of $t^{\rho-2}\varphi(t)$. The result is stated in the form of a theorem which is illustrated by an example.

2. *Theorem.*

If

$$\varphi(s) \underset{m}{\overset{k}{=}} f(t) \tag{2·1}$$

and

$$\Psi(s;\sigma) = G_\sigma \left\{ t^{\rho-2}\varphi(t); s \right\} \tag{2·2}$$

*For the sake of brevity the symbol $\Gamma(a \pm b)$ is used to denote $\Gamma(a+b)\,\Gamma(a-b)$.

than

(2·3) $$\Psi(s;\sigma) = \frac{s^{\rho-\sigma}}{\Gamma(\sigma) * \Gamma(\frac{1}{2}-k\pm m)} \int_0^\infty G^{3\,2}_{2,3}\left(2\,st \,\middle|\, \begin{matrix} k+\frac{3}{4},\ 1-\rho \\ \frac{1}{4}+m,\ \frac{1}{4}-m,\ \sigma-\rho \end{matrix}\right) f(t)\,dt$$

provided the integral is convergent, the Whittaker transform of $|f(t)|$ and generalized Stieltjes transform.

of $|t^{\rho-2}\varphi(t)|$ exist, $R(s) \geqslant s_0 > 0$, $R(\sigma-\rho) > 0$,

$R(\frac{1}{4}\pm m+\rho) > 0$, $R(k\pm m) < \frac{1}{2}$, $R(k+\rho-\sigma-\frac{1}{4}) < 0$ and $|\arg s| < \pi$.

Proof:— Using the integral due to Saxena [5, p. 304 (6)] we have

(2·4) ... $$\frac{s\,\beta^{\rho-\sigma}}{\Gamma(\sigma) * \Gamma(\frac{1}{2}-k\pm m)} G^{3,\,2}_{2,\,3}\left(2s\,\beta \,\middle|\, \begin{matrix} k+\frac{3}{4},\ 1-\rho \\ \frac{1}{4}+m,\ \frac{1}{4}-m,\ \sigma-\rho \end{matrix}\right)$$

$$\underset{m}{\overset{k}{=}}\ t^{\rho-1}(t+\beta)^{-\sigma}$$

where $R(s) \geq s_0 > 0$, $R(\sigma-\rho) > 0$, $R(\frac{1}{4}\pm m+\rho) > 0$,

$R(k\pm m) < \frac{1}{2}$, $R(k+\rho-\sigma-\frac{1}{4}) < 0$ and $|\arg.\beta| < \pi$.

Now, applying (Bose 2, p. 20, R.6):

If

$$\varphi_1(s) \underset{m}{\overset{k}{=}} f_1(t)$$

and

$$\varphi_2(s) \underset{m}{\overset{k}{=}} f_2(t)$$

then

$$\int_0^\infty \varphi_1(u) f_2(u)\, u^{-1}\, du = \int_0^\infty f_1(t)\, \varphi_2(t)\, t^{-1}\, dt.$$

to the relations (2·1) and (2·4), replacing

β by s, we get the result (2·3) as stated above in the theorem.

3. We give below an example illustrating the theorem.

If $\quad f(t) = t^{l-1}\, G^{h,\, r}_{\mu,\, \nu}\left(t^{-1} \middle| \begin{matrix} a_1, \dots, a_\mu \\ b_1, \dots, b_\nu \end{matrix} \right)$

then using the integral due to Bhise (1, p. 74), we have

$$(3\cdot1) \quad \dots \quad \varphi(s) = \frac{s}{{}^*\Gamma\left(\frac{1}{2}-k\pm m\right)}\, G^{k+2,\, r+1}_{\mu+1,\, \nu+2}\left(2 \middle| \begin{matrix} k+\frac{3}{4},\, a_1-l, \dots, a_\mu - l \\ \frac{1}{4}+m,\, \frac{1}{4}-m,\, b_1-l, \dots, b_\nu - l \end{matrix} \right)$$

where $\mu+\nu < 2(h+r)$, $|\arg s| < (h+r-\frac{1}{2}p-\frac{1}{2}q)\pi$,

$R(k\pm m) < \frac{1}{2}$, $R(a_i - 1 \pm m) < 5/4$, $i=1,\dots, r$;

$R(k+1-b_j) > \frac{1}{4}$, $j=1, \dots, h$.

Then, substituting for $f(t)$ in (2.3), applying the transformation to the G-function [3, p. 209 (9)], evaluating the integral with the help of a known integral (Saxena 6, p. 401) and combined with (2.2) we have

$$\Psi(s;\sigma) = \frac{s^{\rho-\sigma}}{\Gamma(\sigma)\, {}^*\Gamma\left(\frac{1}{2}-k\pm m\right)}\, G^{h+3,\, r+2}_{\mu+2,\, \nu+3}$$

$$\left(2s \middle| \begin{matrix} 1-\rho,\, k+\frac{3}{4},\, a_1-l, \dots, a_\mu - l \\ \sigma-\rho,\, \frac{1}{4}+m,\, \frac{1}{4}-m,\, b_1-l, \dots, b_\nu - l \end{matrix} \right) =$$

$$= G_\sigma\left\{ \frac{t^{\rho-1}}{{}^*\Gamma\left(\frac{1}{2}-k\pm m\right)}\, G^{h+2,\, r+1}_{\mu+1,\, \nu+2}\left(2t \middle| \begin{matrix} k+\frac{3}{4},\, a_1-l, \dots, a_\mu - l \\ \frac{1}{4}+m,\, \frac{1}{4}-m,\, b_1-l, \dots, b_\nu - l \end{matrix} \right); s \right\}$$

Replacing $h+2$ by m, $r+1$ by n, $\mu+1$ by p, $\nu+2$ by q,

$k+\frac{3}{4}$ by α_1, a_i-1 by α_{1+i}, $i=1,\dots\mu$; $\frac{1}{4}+m$ by β_1,

$\frac{1}{4}-m$ by β_2, b_j-1 by β_{2+j}, $j=1,\dots,\nu$; we get

$$(3\cdot2) \quad \dots \quad \frac{s^{\rho-\sigma}}{\Gamma(\sigma)}\, G^{m+1,\, n+1}_{p+1,\, q+1}\left(2s \middle| \begin{matrix} 1-\rho,\, \alpha_1, \dots, \alpha_p \\ \sigma-\rho,\, \beta_1, \dots, \beta_q \end{matrix} \right)$$

$$= G_\sigma\left\{ t^{\rho-1}\, G^{m,\, n}_{p,\, q}\left(2t \middle| \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \right); s \right\}$$

where $p+q < 2(m+n)$, $R(\rho+\beta_j) > 0$, $j=1,\dots m$;

$R(\rho-\sigma+\alpha_i) < 1$, $i=1,\ldots n$; $|\arg s| < \pi$.

The result (3·2) tallies with the relation [4, p. 418 (4)] which is in the form of an integral.

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REFERENCES

1. Bhise, V. M., "Some relations involving Meijer's K-transform" *J. Vikram Univ. IV*, pp. 68-77 (1960).

2. Bose, S. K., "A study of the generalized Laplace Integral I" *Bull. Cal. Math. Soc.*, 41, pp. 9-27 (1949).

3. Erdelyi, A., Higher, Transcendental Functions Vol. I McGraw-Hill, (1953).

4. Erdelyi, A., Tables of Integral Transforms Vol. II McGraw-Hill (1954).

5. Saxena, R. K., "Relation between Whittaker transform and modified $x_{v,k,m}$ -transform" *Math. Ann.* 154, 301—306 (1964).

6. Saxena, R. K., "Some theorems on generalized Laplace transform I" *Proc. Nat. Inst. of Sci. India*, 26A, pp. 400-413 (1960).

7. Varma, R. S., "A generalization of Laplace transform" *Current Science*, Vol. 16, pp. 17-18 (1947).

THE SOLAR MOTION AND K-TERM FROM THE RADIAL VELOCITIES OF THE NEW GENERAL CATALOGUE

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SUMMARY

The radial velocities of stars having quoted qualities *a, b, c,* are used to derive the K-term and the coordinates of the solar apex. The stars with proper motions $\geqslant 0''.5$ and or radial velocities exceeding 60 km/sec were excluded. The mean values of K, V_c, α_0, δ_0, are 1.97, 19.04 $\pm$1.16, 265°.6 $\pm$ 2.64, 33.1$\pm$2.22 respectively. The errors are probable errors.

INTRODUCTION

The problem of K-term and solar motion from radial velocities and proper motions has been discussed notably by Plaskett and Pearce (1,2), Smart (3,4), Smart and Green (5), Smart and Tannahill (6,7), Plaskett (8), Tannahill (9,10,11), Ali (12), Walkey (13), Ewart (14), Feast and Thackeray (15). The details of these investigations are given in the papers referred to. In the present paper we have analysed the radial velocities given in the New General Catalogue by Dr R.E. Wilson (16), for the K-term and the solar motion. The method of investigation is as follows :—

The radial velocities with quoted qualities *a, b, c,* were grouped according to the spectra O, B, A, F, G, K, M. Stars of each spectra were divided in two groups-one with positive declinations and one with negative declinations. Each group was then sub-divided into sixteen groups with $1\frac{1}{2}$ hours R. A. so that the mean R. A. for the first group was $0^h\,45^m$ and so on. The means in the positive and negative declinations of the radial velocities were formed. Then the well known equation of condition

$$lx+my+nz+k = \rho$$

was used, l, m, n and ρ have usual meaning. The z-component in the above equation becomes indeterminate. Consequently, assuming the solar velocity as 19.5 km/sec and the equatorial coordinates of the solar apex as $\alpha_0 = 267^\circ.5$ and $\delta_0=32^\circ$ the z-component was found to be$-$10.33. Thus $z \sin \delta$ was eliminated from the equations of condition. The least spuare solution then gives the following values. The weights were given proportional to the number of stars. The second line in table I for each spectra gives the results from stars in the negative declination.

TABLE I

Spectra	N	K	V_o	α_o	δ_o
B	1178	+ 2·26	16·58 ± 1·48	265·8 ± 0·12	38°·4 ± 3·20
	917	+ 1·12	20·77 ± 1·11	268·1 ± 1·51	29·4 ± 2·23
A	1769	+ 2·38	18·04 ± 0·60	268·2 ± 1·57	35·0 ± 1·15
	392	− 1·57	18·12 ± 0·58	265·2 ± 0·24	34·7 ± 0·97
F	1525	+ 3·89	20·16 ± 0·68	268·9 ± 1·99	30·8 ± 1·39
	491	- 1·49	20·70 ± 1·00	268·8 ± 1·93	30·5 ± 1·57
G	1628	+ 1·81	18·37 ± 0·40	263·9 ± 1·03	34·2 ± 0·66
	625	− 0·43	18·27 ± 0·46	266·1 ± 0·30	34·5 ± 0·84
K	1644	+ 0·99	17·97 ± 0·65	261·9 ± 2·33	35·1 ± 1·21
	590	+ 1·10	18·94 ± 0·06	262·5 ± 1·87	33·1 ± 0·00
M	843	− 1·13	19·31 ± 0·13	258·4 ± 4·35	32·3 ± 0·48
	376	+ 4·23	21·25 ± 1·33	268·9 ± 1·99	29·1 ± 2·42

The values of K-though irregular, do not reveal any systematic differences. The negative values of K in the negative declinations are not large enough to reveal a general contraction of the Galaxy. Moreover the values of Vo, α_0, δ_0 are quite in agreement with the accepted values of these quantities given in Allen's Astrophysical Quantities (17). A look at the table reveals that δ_0 is large for positive declinations and small for negative declinations except in the spectra G. The mean values of K with and without regard to sign works out to be 2.22 for +ive, 1.15 for −ive, and 1.97 respectively. The mean values of V_o, α_o, δ_0, are 19·04± 1.16, 265°.6± 2.64, 33°.1±2.22 respectively.

REFERENCES

1&2. Plaskett J. S. & Pearce, J. A., M. N. 94, 679, 1934, publ. D. A. O. Victoria B. C. No 4, 1936.

3&4. Smart, W. M.., M. N. 96, 569, 1936, M. N. 100, 370, 1940.

5. Smart, W. M. & Green, H. E., M. N. 96, 471, 1936.

6&7. Smart, W. M. & Tannahill, T. R., M. N. 98, 563, 1938, M. N. 100, 30, 1940.

8- Plaskett, J. S., M. N., 98, 518, 1938.

9,10&11. Tannahill, T. R., M. N. 112, 3, 1952, M. N. 114, 593, 1954, M. N, 114, 461, 1954.

12. Ali, A., M. N. 101, 325, 1941.

13. Walkey, O. R., M. N. 106, 274, 1946.

14. Ewart, D. G., M. N. 113, 553, 1953.

15. Feast, M. W. & Thackeray, A. D., M. N. 118, 125, 1958.

16. Wilson, R. E. Carnegie Institution of Washington Publ. 601, 1953.

16. Allen, C. W., Astrophysical Quantities University of London, 1955.

STEADY FLOW OF A VISCOUS INCOMPRESSIBLE FLUID THROUGH AN ELASTIC TUBE WITH SUCTION AT THE BOUNDARY

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ABSTRACT

Steady laminar flow of a viscous incompressible fluid through an elastic tube with porous walls is considered. A normal suction is applied at the boundry. It is shown that as a result of this suction the tube expands in the direction of the flow if the fluid viscosity is sufficiently small. The suction parameter also increases in the same direction, from its value at the entry of the tube which is greater than 2.

Both the radial and the axial velocities decrease all along the tube. With a continuous decrease in velocity the Reynolds number also decreases, so that the flow is likely to remain laminar if it is so at the entry. There is an increase in pressure as the fluid moves along and this at some stage may cause backward flow at the boundary.

INTRODUCTION

In a recont paper R. G. Choudary and K. P. Sinha [1] have discussed the steady laminar flow of a viscous incompressible fluid through a circular tube of constant radius with porous walls. They have predicted that with a small normal outward suction at the boundary the flow expands as the fluid moves along the axis of the tube. An attempt has been made in the present note to prove this result and it is shown that this is true for a fluid of very small viscosity. We consider the flow through an elastic narrow tube of circular cross section. It is shown that the radius of the tube increases as the distance along the axis of the tube increases if the fluid viscosity is sufficiently small. If the viscosity is not small enough we get the usual convergent glow.

(r, ϕ, z) are the cylindrical coordinates. The velocity components in the r and z directions are denoted respectively by v and u. The motion is symmetrical about the axis of the tube and the fluid pressure p is assumed to be uniform over any cross-section of the tube. The equations of motion are

$$\rho \left(v \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial z} \right) = \mu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{1}$$

$$\rho \left(v \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z} \right) = - \frac{dp}{dz} + \mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right) \tag{2}$$

The equation of continuity is

$$\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial u}{\partial z} = 0 \tag{3}$$

In the usual notation the stress-strain relations are [2]

$$P_{ij} = - p\,\delta_{ij} + 2\mu\, e_{ij} - \frac{2}{3}\,\mu\,\delta_{ij}\, e_{kk} \tag{4}$$

R (z) is taken as the variable radius of the tube. If E denotes the Young's modulus and δ is the thickness of the wall, it is easy to verify that [3]

$$P_{rr} = E\,\delta\left(\frac{1}{L} - \frac{1}{R}\right) \tag{5}$$

where L is the radius of the unstretched tube and is consquently taken as constant.

1. For the present we neglect the terms $v\,\dfrac{\partial v}{\partial r}$, $u\,\dfrac{\partial v}{\partial z}$, $\dfrac{\partial^2 v}{\partial z^2}$ in (1) and give the justification for the same latter. With this (1) gives

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\,\frac{\partial v}{\partial r} - \frac{v}{r^2} = 0 \tag{6}$$

The solution of this with $v = v_0$ on the boundary $r = R\,(z)$ and finite on the axis is

$$v = \frac{v_0}{R\,(z)}\, r \tag{7}$$

The equation (3) now gives

$$\frac{\partial u}{\partial z} = -\,2\,\frac{v_0}{R\,(z)} \tag{8}$$

Substituting (7) and (8) in (2) we have

$$\rho\left[u\left(\frac{-2v_0}{R}\right) + r\left(\frac{v_0}{R}\right)\frac{\partial u}{\partial r}\right]$$

$$= -\,\frac{dp}{dz} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial \mu}{\partial r} + \frac{2v_0}{R^2}\,\frac{dR}{az}$$

where we have written $R = R(z)$. Making the substitution

$\eta = \frac{r}{R}$ the above equation becomes

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \mu}{\partial \eta} - \left(\frac{v_0 R}{\nu}\right)\eta\frac{\partial u}{\partial \eta} + \left(\frac{2v_0 R}{\nu}\right)u$$

$$-\left[\frac{R^2}{\mu}\frac{dp}{dz} - 2v_0\frac{dR}{dz}\right] = 0 \tag{9}$$

The solution of this equation with $u = 0$ for $\eta = 1$ and

$\frac{\partial u}{\partial \eta} = 0$ for $\eta = 0$ is

$$u = -\left[\frac{R^2\frac{dp}{az} - 2\mu v_0\frac{dR}{dz}}{\alpha\mu\left(1 - \frac{v_0 R}{\nu}\right)}\right] \tag{10}$$

From (4) we have

$$P_{rr} = \frac{dp}{dz} + 2\mu\left(\frac{v_0}{R^2}\right)\frac{dR}{dz}$$

Differentiating the above with respect to z

$$\frac{\partial P_{rr}}{\partial z} = \frac{dp}{dz} + 2\mu\left(\frac{v_0}{R^2}\right)\frac{dR}{dz} \tag{11}$$

Differentiating (5) with respect to z aubstituting in (11) we get

$$(E\delta - 2\mu v_0)\frac{dR}{dz} = R^2\frac{dp}{dz} \tag{12}$$

Substituting this in (10) we obtain

$$u = -\left[\frac{E\delta - \alpha\mu v_0}{\alpha\mu\left(1 - \frac{v_0 R}{2\nu}\right)}\right]\frac{dR}{dz}(1 - \eta^2) \tag{13}$$

We thus observe that the axial velocity has the usual parabolic peofile.

Denoting by Q the volume of the fluid flowing across a cross-section per unit time we have

$$Q = 2\pi \int_0^{R(z)} r\,u\,dr = 2\pi R^2 \int_0^1 \eta\,u\,d\eta$$

Substituting from (13) integrating we get

$$Q = -\pi R^2 \frac{dR}{dz}\left[\frac{E\,\delta - \propto \mu v_o}{8\mu\left(1 - \frac{v_o R}{2\nu}\right)}\right] \tag{14}$$

Ie we put $Q = \pi R^2 V$ where V is the average velocity over a cross-section we have

$$\frac{dR}{\partial z} - \left[\frac{\propto \mu v_o V}{\nu(E\,\delta - \propto \mu v_o)}\right] R = -\frac{8\mu V}{E\,\delta - \propto \mu v_o} \tag{15}$$

The above equation is integrated to

$$R = \left(R_o - \frac{2v}{v_o}\right) e^{\alpha z} + \frac{2v}{v_o} \tag{16}$$

where V, is assumed to be constant for a slowly expanding tube [it will be shown later $\frac{dR}{dz}$ is small] and

$$\alpha = \frac{\propto \mu v_o V}{\nu(E\,\delta - \propto \mu v_o)} \tag{17}$$

and R_o is the value of R at $z = z_o$

(16) can also be written as

$$R = R_o + \frac{2\nu}{v_o}\left(\frac{R_o v_o}{2\nu} - 1\right)(\alpha z) +$$

$$+ \frac{2v}{v^o}\left(\frac{R_o v_o}{2\nu} - 1\right)\frac{(\alpha z)^2}{2!} + \frac{2v}{v_o}\left(\frac{R_o v_o}{2v} - 1\right)\frac{(\alpha z)^3}{3!} + \cdots \tag{18}$$

We observe from (18) that R increases with z if $\frac{R_0 v_0}{2\nu} > 1$ or $\mu < \frac{R_0 v_0 \rho}{2}$.

Thus for a given small v_0, expansion is possible when the fluid viscosity is sufficiently small. If $\sigma(z) = \frac{R v_0}{\nu}$ denotes the suction parameter (18) can also be written as

$$\frac{\sigma(z)}{2} = \frac{\sigma_0}{2} + \left(\frac{\sigma_0}{2} - 1 \right)(\alpha z) + + \left(\frac{\sigma_0}{2} - 1 \right) \frac{(\alpha z)^2}{2!} + \left(\frac{\sigma_0}{2} - 1 \right) \frac{(\alpha z)^3}{3!} + \ldots \tag{19}$$

where σ_0 is the value of $\sigma(z)$ at the entry. There is an expansion of the flow for $\sigma_0 > 2$. $\sigma(z)$ increases all along the pipe and its value at the entry is > 2. With this it is seen from (13) that the axial velocity can never be infinity as in the case of the flow through a tube of constant radius [1].

Justification of the terms neglected:—

We see from (7) that

$$\frac{\partial v}{\partial z} = -\frac{r v_0}{R^2} \frac{dR}{dz}, \quad \frac{\partial^2 v}{\partial z^2} = -\frac{r v_0}{R^2} \frac{d^2R}{dz^2} + \frac{2r v_0}{R^3} \left(\frac{dR}{dz} \right)^3$$

For a small v_0 we observe from (18) that $\left[\text{since } \mu < \frac{R_0 v_0 \rho}{2} \right]$

$$\frac{dR}{dz} = O(v_0), \quad \frac{d^2R}{dz^2} = O(v_0^2)$$

Thus for small v_0, $\frac{\partial^2 v}{\partial z^2} = O(v_0^2)$ and similarly

$v \frac{\partial v}{\partial z}$ and $u \frac{\partial v}{\partial z}$ are each of order $O(v_0^3)$.

where as

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} = O(v_0)$$

Thus for a small v_0 (2) simplies to (6). No terms are neglected in (1).

It is easy to verify from (7), (12) and (14) v, u, and Q all decrease as z increases. The Reynolds number also decreases and the flow is likely to remain laminar if it is so at the entry. There is an increase in the fluid pressure all along the tube and this is superimposed on the pressure drop due to friction at the boundary. If the resultant pressure drop in the direction of the flow becomes negative there is a possibility that the fluid near the walls starts moving back.

REFERENCES

1. Choudhary R. C. and Sinhak. P., *Nat. Acad. Sci. India, XXXIV*, 203 (1964) part (2).
2. Pai, S. I., "Viscous flow theory " Vol. 1, (1956).
3. Love A. E. N., "A treatise on the mathematical theory of elastity (1952).

STUDIES ON TUNGSTATES OF THORIUM AND EFFECT OF AGEING

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ABSTRACT

The composition of thorium tungstates, obtained by the addition of thorium nitrate to a progressively acidified solution of sodium tungstate, has been studied by conductometric and quantitative methods. The conclusions drawn are in agreement with those arrived at by the potentiometric studies of similar systems, described earlier.[1]

EXPERIMENTAL

All solutions were prepared with A. R. quality reagents observing usual precautions. Electrical conductance measurements were made with a LBR model B177 measuring bridge operated on 220V/50 cycles a. c. mains having a magic eye electronic detector. The conductance measurements were made at $25^{\circ} \pm 0.5^{\circ}C$. The effect of atmospheric carbon dioxide was avoided by keeping the solutions in test-tubes completely filled and well stoppered.

10 ml of 0.1M sodium tungstate solution were taken in different pyrex glass test-tubes. Required quantity of 0.25M nitric acid was added to each to have, theoretically, tungstate ions of a particular composition[1]. 0.1M thorium nitrate solution was added from a micro-burette in regularly increasing instalments. The electrical conductance of these solutions, kept out of contact from atomospheric carbon dioxide, was noted at start, after 24 hrs., 3 days and 6 days. Titrations were performed with 0.067M and 0.05M sodium tungstate solutions also and the results were, in general, similar.

In the quantitative studies the metal and tungstate ions, left in the supernatant liquid, were estimated by usual methods after precipitation had taken place in systems similar to those mentioned above and maintained at the same temperature.

DISCUSSION

The results of the present studies confirm those arrived at from the potentiometric studies[1] that five different tungstates of thorium can be prepared by metathetic methods, using the interaction between sodium tungstate solution (containing varying amounts of nitric acid to form sodium tungstates in different degrees of polymerisation) and thorium nitrate.

Normal, di- and tritungstate of thorium, ThO_2. $2WO_3$, ThO_2 $4WO_3$ and $ThO_2.6WO_3$ respectively, are formed by direct interaction and ageing has no pronounced effect on their composition.

The paratungstate of thorium, $3ThO_2.14WO_3$ formed by direct interaction between thorium nitrate and the corresponding sodium tungstate, has a tendency to hydrolyse on ageing forming thorium tritungstate.

$$3(3ThO_2.14WO_3) + 4H_2O = 7(ThO_2.6WO_3) + 2Th(OH)_4$$

It is noteworthy that, unlike some metals[2], in the beginning thorium paratungstate of composition $3ThO_2.14WO_3$, is formed whether the $HNO_3 : Na_2WO_4$ ratio is 1.14 or 1.15.

Sodium tetratungstate precipitates little thorium tritungstate at first which on ageing gets converted into the tetratungstate.

$$2(Na_2O.4WO_3) + Th(NO_3)_4 + H_2O = ThO_2.6WO_3 + 4NaNO_3 + 2H_2WO_4$$
$$ThO_2.6WO_3 + 2H_2WO_4 = ThO_2.8WO_3 + H_2O$$

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REFERENCES

1. Shivahare, G. C., *Proc. Nat. Acad. Sci., India* 32A, 127 (1962)
2. Shivahare, G. C., *Naturwiss.*. 17, 406 (1964)

CERTAIN THEOREMS ON GENERALISED LAPLACE TRANSFORMS

By

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INTRODUCTION

The well-known laplace Transform

$$\phi(p)=p\int_0^{\infty} e^{-pt} f(t)\,dt \tag{1·1}$$

has been generalised by Meijer (3) in the form

$$\phi(p)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} p\int_0^{\infty} (pt)^{\frac{1}{2}} K_v(pt) f(t)\,dt \tag{1·2}$$

and by Mainra (5) in the form

$$\phi(p)=\int_0^{\infty} (pt)^{\lambda-\frac{1}{2}} e^{-\frac{1}{2}pt} W_{k+\frac{1}{2},m}(pt)\, f(t)\,dt \tag{1·3}$$

which reduces to another generalisation of Meijer (4)

when $\lambda=-k-\frac{1}{2}$ and to Varma transform (8) when $\lambda=m$.

The object of this paper is to prove certain theorems depicting the relation of (1.2) or (1.3) with either Y transform or H transform given by Titchmarsh (7) such that if

$$h(p)=\int_0^{\infty} (pt)^{\frac{1}{2}} Y_v(pt) f(t)\,dt \tag{1·4}$$

then

$$f(p)=\int_0^{\infty} (pt)^{\frac{1}{2}} H_v(pt)\, h(t)\,dt \tag{1·5}$$

We shall represent the equations (1.2) to (1.5) respectively as

$$\phi(p) \doteq f(t)$$

$$\phi(p) \underset{v}{\overset{K}{=}} f(t)$$

$$\phi(p) \overset{W}{\underset{\lambda,\, k,\, m}{=}} f(t)$$

$$\phi(p) \overset{\Upsilon}{\underset{v}{=}} f(t)$$

and

$$\phi(p) \overset{H}{\underset{v}{=}} f(t)$$

Also throughout this paper we shall use the following abbreviations for the sequences mentioned against them :—

$$\Delta(s,\, a) = \frac{a}{s},\ \frac{a+1}{s},\ \ldots\ldots\ldots\ldots,\ \frac{a+s-1}{s}$$

$$\Delta(s,\, a \pm b) = \frac{a+b}{s},\ \ldots\ldots,\ \frac{a+b+s-1}{s},\ \frac{a-b}{s},\ \frac{a-b+1}{s},\ \ldots\ldots\ldots,\ \frac{a-b+s-1}{s}$$

2. *Theorem 1.* If $x^{\lambda+\frac{1}{2}} f(x^{r/s})$ and $x^{v+3/2}$ both belong to the class L (O, ∞), R $[\lambda \pm \mu + (v+3/2)\, r/s + 2] > O$

$$\psi(p) \overset{k}{\underset{\mu}{=}} x^{\lambda+\frac{1}{2}} f(x^{r/s})$$

and

$$\phi(p) \overset{\Upsilon}{\underset{v}{=}} f(x)$$

then

$$\psi(p) = (2\pi)^{\frac{1}{2}-r} \times (2r)^{\lambda+1} \times (2\, s)^{\frac{1}{2}} \times$$

$$\times \int_0^\infty G^{s,\, s+2r}_{s+2r,\, 3s}\left[\frac{(2r)^{2r}}{(2s)^{2s}}\ \frac{\Upsilon^{2s}}{p^{2r}}\ \middle|\ \begin{matrix} \Delta\left(s, \frac{3}{4}+\frac{v}{2}\right),\ \Delta\left(r, -\frac{1}{2}\lambda \pm \frac{1}{2}\mu\right) \\ \Delta\left(s, \frac{3}{4}+\frac{v}{2}\right),\ \Delta\left(s, \frac{1}{4} \pm \frac{1}{2}v\right) \end{matrix}\right] \phi(y)\, dy \qquad (2.1)$$

Proof: By virtue of (1.5)

$$f(x^{r/s}) = \int_0^\infty (y\, x^{r/s})^{\frac{1}{2}}\ \mathrm{H}v\,(x^{r/s} y)\ \phi(y)\, dy$$

and therefore

$$\phi(p) = p\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty (px)^{\frac{1}{2}}\ \mathrm{K}v\,(px)\ x^{\lambda+\frac{1}{2}} \left\{ \int_0^\infty (y x^{r/s})^{\frac{1}{2}}\ \mathrm{H}v\,(x^{r/s} y)\ \phi \right\} (y)\, dy \,\Big\}\, dx$$

Now changing the order of integration (which is permissible under the conditions stated) and evaluating the inner integral with the help of the result given by Saxena (6) after applying the transformations (1, p. 38))

$$x^{\mu}\, H_{v}(x) = 2^{\mu}\; G^{11}_{13}\left[\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2}+\tfrac{1}{2}v+\tfrac{1}{2}\mu \\ \tfrac{1}{2}+\tfrac{1}{2}v+\tfrac{1}{2}\mu,\ \tfrac{1}{2}\mu+\tfrac{1}{2}v,\ \tfrac{1}{2}\mu-\tfrac{1}{2}v \end{matrix}\right] \tag{2.2}$$

and

$$x^{\mu} K_{\mu}\;(x) = 2^{-1}\; G^{40}_{04}\left[\tfrac{1}{4}x^2 \,\middle|\, \tfrac{1}{2}\mu+\tfrac{1}{2}v,\ \tfrac{1}{2}\mu-\tfrac{1}{2}v\right] \tag{2.3}$$

we get (2.1) (2.1)

Below are given some of the particular cases of the above theorem :

Theorem (1.1) Putting $r=n$, $s=1$ the theorem reduces to one that if $x^{\lambda+\frac{1}{2}} f(x^{n})$ and $x^{v+3/2} \times \phi(x)$ both ϵ L (O, ∞) R $[\lambda\pm\mu+n\,(v+3/2)+2]>O$

$$\psi(p) \underset{\mu}{\overset{K}{==}} x^{\lambda+\frac{1}{2}} f(x^{n})$$

and
$$\phi(p) \underset{v}{\overset{Y}{=}} f(x)$$

then $\psi(p)=2^{\lambda-n+2} \times \pi^{\frac{1}{2}-n} \times n^{\lambda+1} \times p^{-\lambda-\frac{1}{2}} \times$

$$\times\int_0^{\infty} G^{1,2n+1}_{2n+1,3}\left[\frac{(2n)^{2n}}{4} \times \frac{y^2}{p^{2n}} \,\middle|\, \begin{matrix} \tfrac{3}{4}+\tfrac{v}{2},\ \Delta\,(n,\ \tfrac{1}{2}\lambda\pm\tfrac{1}{2}\mu) \\ \tfrac{3}{4}+\tfrac{v}{2},\ \tfrac{1}{4}+\tfrac{1}{2}v,\ \tfrac{1}{4}-\tfrac{1}{2}v \end{matrix}\right] \phi(y)\,dy \tag{2.4}$$

Theorem 1.2 : If we take $r=1$, $s=n$ then it reduces to that when $x^{\lambda+\frac{1}{2}} f(x^{1/n})$, $x^{v+3/2}\ \phi(x)$ both ϵ L (O, ∞), R $[2+\lambda\pm\mu+1/n\,(v+3/2)]>O$

$$\psi(p) \underset{\mu}{\overset{K}{=}} x^{\lambda+\frac{1}{2}} f(x^{1/n})$$

and

$$\phi(p) \underset{v}{\overset{Y}{=}} f(x)$$

then
$$\psi(p)=2^{\lambda+1}\times p^{-\lambda-\frac{1}{2}}\times\left(\frac{n}{\pi}\right)^{\frac{1}{2}}\times$$

$$\times\int_0^{\infty} G^{n,\,n+2}_{n+2,\,3n}\left[\frac{4y^{2n}}{(2n)^{2n}p^2} \,\middle|\, \begin{matrix} \Delta\,(n,\ \tfrac{3}{4}+\tfrac{1}{2}v),\ -\tfrac{1}{2}\lambda-\tfrac{1}{2}\mu,\ -\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu \\ \Delta\,(n,\ \tfrac{3}{4}+\tfrac{1}{2}v),\ \Delta\,(n,\ \tfrac{1}{4}\pm\tfrac{1}{2}v) \end{matrix}\right] \phi(y)dy \tag{2.5}$$

Theorem 1.3 :—When we take $r=1$ and $s=2$ we get that if

$$\psi(p) \stackrel{K}{\underset{\mu}{=}} x^{\lambda+\frac{1}{2}} f(x^{\frac{1}{2}})$$

and

$$\phi(p) \stackrel{\Upsilon}{\underset{v}{=}} f(x)$$

then $\psi(p) = 2^{\lambda+3/2} \times p^{-\lambda-\frac{1}{2}} \times \pi^{-\frac{1}{2}} \times$

$$\times \int_0^\infty G^{2\,4}_{4\,6}\left[\frac{y^4}{64p^2}\left|\begin{matrix}\Delta(2, \frac{3}{4}+\frac{1}{2}v), -\frac{1}{2}\lambda-\frac{1}{2}\mu, -\frac{1}{2}\lambda+\frac{1}{2}\mu\\ \Delta(2, \frac{3}{4}+\frac{1}{2}v), \Delta(2, \frac{1}{4}\pm\frac{1}{2}v)\end{matrix}\right.\right] \phi(y)\, dy \tag{2.6}$$

provided that $x^{\lambda+\frac{1}{2}} f(x^{\frac{1}{2}})$ and $x^{v+3/2}\phi(x)$ both $\in L(O, \infty)$ and $R[2\lambda \pm 2\mu + v + 7/2] > O$.

Theorem 1.4 :—Further if we take $\mu=\frac{1}{2}$ in theorem (1.3), then by virtue of the formulas :—

$$G^{\alpha\,\beta}_{\gamma\,\delta}\left(x\left|\begin{matrix}a_1,\ldots\ldots, a_\gamma\\ b_1,\ldots\ldots, b_\delta\end{matrix}\right.\right) = (2\pi)^{(1-s)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)} \times s^{\Sigma b_j - \Sigma a_j + \frac{1}{2}\gamma - \frac{1}{2}\delta + 1} \times$$

$$\times G^{s\alpha,\, s\beta}_{s\gamma,\, s\delta}\left(\begin{matrix} s^{s(\gamma-\delta)}\\ x^s\end{matrix}\left|\begin{matrix}\Delta(s, a_1), \ldots \Delta(s, a_\gamma)\\ \Delta(s, b_1), \ldots \Delta(s, b_\delta)\end{matrix}\right.\right) \tag{2.7}$$

and

$$G^{1\,p}_{p\,q}\left(x\left|\begin{matrix}a_1,\ldots\ldots, a_p\\ b_1,\ldots\ldots, b_q\end{matrix}\right.\right) = \frac{\prod_{j=1}^{p}\Gamma(1+b_1-a_j)}{\prod_{j=2}^{q}\Gamma(1+b_1-b_j)}\, x^{b_1}$$

$${}_pF_{q-1}\left(\begin{matrix}1+b_1-a_1,\ldots\ldots, 1+b_1-a_p,\\ 1+b_1-b_2,\ldots\ldots, 1+b_1-b_q,\end{matrix} -x\right)_{p\leq q} \tag{2.8}$$

it gives us that if $x^{\lambda+\frac{1}{2}} f(x^{\frac{1}{2}}), x^{v+3/2}\phi(x) \in L(O, \infty)$, R $[2\lambda+v+5/2] > O$

$$\psi(p) \doteqdot x^{\lambda+\frac{1}{2}} f(x^{\frac{1}{2}})$$

and

$$\phi(p) \stackrel{\Upsilon}{\underset{v}{=}} f(x)$$

then $\psi(p) = \dfrac{p^{-\lambda-\frac{1}{2}v-5/4}}{\sqrt{\pi}} \times 2^{-v} \times \dfrac{\Gamma(\lambda+\frac{1}{2}v+9/4)}{\Gamma(v+3/2)} \times$

$$\times \int_0^\infty \phi(y) \times y^{v+3/2} \, {}_2F_2\left[\begin{matrix} 1, \lambda+\frac{1}{2}v+9/4, \\ 3/2, v+3/2, \end{matrix} \; -\frac{y^2}{4p^2}\right] dy \tag{2.9}$$

If we take $r=s$, then by virtue of the formulas (2.7) and (2.8) we get the theorem given by C. B. L. Varma (9).

We give below some more theorems Proof of all these theorems are on the same lines as that of first. Transformations applied may be either from amongst (2.2) and (2.3) or from

$$e^{-\frac{1}{2}x} W_{k,m}(x) = \pi^{-\frac{1}{2}} \times x^{\frac{1}{2}} \times 2^{K-\frac{1}{2}} \times G_{24}^{40}\left(\frac{x^2}{4} \middle| \begin{matrix} \frac{1}{4}-\frac{1}{2}K, \frac{3}{4}-\frac{1}{2}K \\ \frac{1}{2}+\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}m, \frac{1}{2}m, -\frac{1}{2}m \end{matrix}\right) \tag{2.10}$$

and

$$x^{\mu} \Upsilon_v(x) = 2^{\mu} \, G_{13}^{20}\left(\frac{x^2}{4} \middle| \begin{matrix} \frac{1}{2}\mu-\frac{1}{2}v-\frac{1}{2} \\ \frac{1}{2}\mu-\frac{1}{2}v, \frac{1}{2}\mu+\frac{1}{2}v, \frac{1}{2}\mu-\frac{1}{2}v-\frac{1}{2} \end{matrix}\right) \tag{2.11}$$

as the case may be

Theorem 2 : If $x^{\lambda+\frac{1}{2}} f(x^{r/s})$, $x^{v+\frac{1}{2}}\phi(x)$ both $\in L(O, \infty)$ $R[\lambda\pm\mu+(\frac{1}{2}\pm v)r/s+2]>O$

$$\psi(p) \overset{K}{\underset{\mu}{=}} x^{\lambda+\frac{1}{2}} f(x^{r/s})$$

and

$$\phi(p) \overset{H}{\underset{v}{=}} f(x)$$

then $\psi(p) = (2\pi)^{\frac{1}{2}-r} \times (2r)^{\lambda+1} \times (2s)^{\frac{1}{2}} \times p^{-\frac{1}{2}-\lambda} \times$

$$\times \int_0^\infty G_{s+2r, 3s}^{2s, 2r}\left[\frac{(2r)^{2r}}{(2s)^{2s}} \cdot \frac{y^{2s}}{p^{2r}} \middle| \begin{matrix} \Delta\left(r, \frac{-\lambda-\mu}{2}\right), \Delta\left(r, \frac{-\lambda+\mu}{2}\right), \Delta(s, -\frac{1}{4}-\frac{1}{2}v) \\ \Delta(s, \frac{1}{4}-\frac{1}{2}v), \Delta(s, \frac{1}{4}+\frac{1}{2}v), \Delta(s, -\frac{1}{4}-\frac{1}{2}v) \end{matrix}\right] \phi(y)\, dy \tag{3.1}$$

Below are given some corollaries of the above mentioned theorem :

Corollary 1 Putting $r=n$, $s=1$ it reduces to that if

$x^{\lambda+\frac{1}{2}} f(x^n)$ and $x^{v+\frac{1}{2}}\phi(x)$ both $\in L(O, \infty)$ $R[\lambda\pm\mu+(\frac{1}{2}\pm v)n+2]>O$

$$\psi(p) \overset{K}{\underset{\mu}{=}} x^{\lambda+\frac{1}{2}} f(x^n)$$

and

$$\phi(p) \overset{H}{\underset{v}{=}} f(x)$$

then $\psi(p) = 2^{\lambda-n+2} \times n^{\lambda+1} \times \pi^{\frac{1}{2}-n} \times p^{-\frac{1}{2}-\lambda}$

$$\times \int_0^\infty G^{2,\,2n}_{1+2n,\,3}\left[\frac{(2n)^{2n}}{4}\cdot\frac{y^2}{p^{2n}}\,\middle|\,\begin{matrix}\Delta\left(n, \frac{-\lambda-\mu}{2}\right), \Delta\left(n, \frac{-\lambda+\mu}{2}\right), -\frac{1}{4}-\frac{1}{2}v \\ \frac{1}{4}-\frac{1}{2}v, \frac{1}{4}+\frac{1}{2}v, -\frac{1}{4}-\frac{1}{2}v\end{matrix}\right]\phi(y)\,dy \tag{3.2}$$

Corollary 2 : When we put $r=1$, $s=1$ in theorem 2, then it reduces to that if

$$\psi(p) \overset{K}{\underset{\mu}{=}} x^{\lambda+\frac{1}{2}} f(x)$$

and

$$\phi(p) \overset{H}{\underset{v}{=}} f(x)$$

then $\psi(p) = -\dfrac{2^{\lambda+1}}{\pi^{3/2}} \times \Gamma(v)\,\Gamma(5/4-\frac{1}{2}v \pm\frac{1}{2}\mu+\frac{1}{2}\lambda) \times p^{-\frac{1}{2}-\lambda+v-\frac{1}{2}} \times$

$$\times \int_0^\infty y^{\frac{1}{2}-v} \times {}_3F_2\left(\begin{matrix}5/4-\frac{1}{2}v+\frac{1}{2}\lambda+\frac{1}{2}\mu,\ 5/4-\frac{1}{2}v+\frac{1}{2}\lambda-\frac{1}{2}\mu,\ 3/2.,\ -\frac{y^2}{p^2} \\ 1-v,\ 3/2.,\end{matrix}\right)\phi(y)\,dy \tag{3.3}$$

where $x^{\lambda+\frac{1}{2}} f(x)$, $x^{v+\frac{1}{2}}\phi(x)$ both $\in L(O, \infty)$ and $R[\lambda\pm\mu\pm v+5/2] > O$

Corollary 3 : While if we put $r=1$, $s=n$ it reduces to that if

$$\psi(p) \overset{K}{\underset{\mu}{=}} x^{\lambda+\frac{1}{2}} f(x^{1/n})$$

and

$$\phi(p) \overset{H}{\underset{v}{=}} f(x)$$

then

$$\psi(p) = 2^{\lambda+1} \times \left(\frac{n}{\pi}\right)^{\frac{1}{2}} \times p^{-\frac{1}{2}-\lambda} \times$$

$$\times \int_0^\infty G^{2n,\,2}_{n+2,\,3n}\left[\frac{4}{(2n)^{2n}}\cdot\frac{y^{2n}}{p^2}\,\middle|\,\begin{matrix}\frac{-\lambda-\mu}{2}, \frac{-\lambda+\mu}{2}, \Delta(n, -\frac{1}{4}-\frac{1}{2}v) \\ \Delta(n, \frac{1}{4}-\frac{1}{2}v), \Delta(n, \frac{1}{4}+\frac{1}{2}v), \Delta(n, -\frac{1}{4}-\frac{1}{2}v)\end{matrix}\right]\phi(y)\,dy \tag{3.4}$$

for $R[\lambda\pm\mu+(\frac{1}{2}\pm v)\ 1/n+2] > O$ and if $x^{\lambda+\frac{1}{2}} f(x^{1/n})$, $x^{v+\frac{1}{2}}\phi(x)$ both $\in L(O,\infty)$

Corollary 4 : Taking $n=2$ in the above Corollary, we have that if $x^{\lambda+\frac{1}{2}} f(\sqrt{x})$ and $x^{v+\frac{1}{2}} \phi(x)$ belong to L (O, ∞) R $[2\lambda \pm 2\mu \pm v + 5/2] > O$ and if

$$\psi(p) \underset{\mu}{\overset{\mathrm{K}}{=}} x^{\lambda+\frac{1}{2}} f(\sqrt{x})$$

and

$$\phi(p) \underset{v}{\overset{\mathrm{H}}{=}} f(x)$$

then

$$\psi(p) = \frac{2^{\lambda+3/2}}{\sqrt{\pi}} \times p^{-\frac{1}{2}-\lambda} \times$$

$$\times \int_0^\infty G^{42}_{46}\left[\frac{y^4}{64p^2} \middle| \begin{matrix} \frac{-\lambda-\mu}{2}, \frac{-\lambda+\mu}{2}, \frac{-\frac{1}{4}-\frac{1}{2}v}{2}, \frac{\frac{3}{4}-\frac{1}{2}v}{2}, \\ \frac{\frac{1}{4}-\frac{1}{2}v}{2}, \frac{5/4-\frac{1}{2}v}{2}, \frac{\frac{1}{4}+\frac{1}{2}v}{2}, \frac{5/4+\frac{1}{2}v}{2}, \frac{-\frac{1}{4}-\frac{1}{2}v}{2}, \frac{\frac{3}{4}-\frac{1}{2}v}{2} \end{matrix}\right] \phi(y)\, dy \tag{3.5}$$

Theorem 3 : If $f(x^{r/s})$, $\phi(x)$ both $\in$ L (O, ∞) R $[\frac{1}{2} \pm v] > O$ R $[\lambda \pm m + r/s(\frac{1}{2} \pm v) + 3/2] > O$

$$\psi(p) \xrightarrow[\lambda, k, m]{\mathrm{W}} f(x^{r/s})$$

and

$$\phi(p) \underset{v}{\overset{\mathrm{H}}{=}} f(x)$$

then

$$\psi(p) = (2\pi)^{\frac{1}{2}-r} \times (2s)^{\frac{1}{2}} \times (2r)^{\frac{1}{2}} \times$$

$$\times \int_0^\infty G^{2s,\, 4r}_{s+4r,\, 3s+2r}\left[\frac{(2r)^{2r}}{(2s)^{2s}} \cdot \frac{y^{2s}}{p^{2r}} \middle| \begin{matrix} \Delta(2r, -\frac{1}{2}-(\lambda+m)), \Delta(2r, -\frac{1}{2}-(\lambda-m)), \Delta(s, -\frac{1}{4}-\frac{1}{2}v), \\ \Delta(s, \frac{1}{4}-\frac{1}{2}v), \Delta(s, \frac{1}{4}+\frac{1}{2}v), \Delta(2r, -2\lambda), \Delta(s, -\frac{1}{4}-\frac{1}{2}v) \end{matrix}\right] \phi(y)\, dy \tag{4.1}$$

Particular cases :

Theorem 3.1 :—If $f(x)$ and $\phi(x)$ belong to L (O, ∞)

also

$$\psi(p) \xrightarrow[\lambda, k, m]{\mathrm{W}} f(x)$$

and

$$\phi(p) \overset{H}{\underset{v}{=}} f(x)$$

then

$$\psi(p) = \frac{\sqrt{2}}{\sqrt{\pi}} \times$$

$$\times \int_0^\infty G_{55}^{24} \left[\frac{y^2}{p^2} \middle| \begin{matrix} \Delta(2, -\tfrac{1}{2}-(\lambda+m)), \Delta(2, -\tfrac{1}{2}-(\lambda-m)), -\tfrac{1}{4}-\tfrac{1}{2}v \\ \tfrac{1}{4}-\tfrac{1}{2}v, \tfrac{1}{4}+\tfrac{1}{2}v, \Delta(2, -2\lambda), -\tfrac{1}{4}-\tfrac{1}{2}v \end{matrix} \right] \phi(y)\, dy \qquad (4{\cdot}2)$$

The above result is obtained by taking $r=s$ in the above stated theorem

Theorem 3.2 : Putting $r=n$ and $s=1$ in the theorem 3, we have that if

$$\psi(p) \underset{\lambda,k,m}{\overset{W}{=}} f(x^n)$$

and

$$\phi(p) \overset{H}{\underset{v}{=}} f(x)$$

then

$$\psi(p) = 2^{3/2-n} \times \pi^{\frac{1}{2}-n} \times \sqrt{n} \times$$

$$\times \int_0^\infty G_{1+4n,\ 3+2n}^{2,4n} \left[\frac{(2n)^{2n}}{4} \cdot \frac{y^2}{p^{2n}} \middle| \begin{matrix} \Delta(2n, -\tfrac{1}{2}-(\lambda+m)), \Delta(2n, -\tfrac{1}{2}-(\lambda-m)), -\tfrac{1}{4}-\tfrac{1}{2}v \\ \tfrac{1}{4}-\tfrac{1}{2}v, \tfrac{1}{4}+\tfrac{1}{2}v, \Delta(2n, -2\lambda), -\tfrac{1}{4}-\tfrac{1}{2}v \end{matrix} \right] \phi(y)\, dy \qquad (4{\cdot}3)$$

provided that $R[\lambda \pm m + (\tfrac{1}{2} \pm v)\ n + 3/2] > O$, $R[\tfrac{1}{2} \pm v] > O$ and $f(x^n)$, $\phi(x)$ both $\in L(O, \infty)$

Thorem 3.3 :—Instead if we put $r=1$ and $s=n$ it gives that if

$$\psi(p) \underset{\lambda,k,m}{\overset{W}{=}} f(x^{1/n})$$

and

$$\phi(p) \overset{H}{\underset{v}{=}} f(x)$$

then

$$\psi(p) = \frac{\sqrt{2} \cdot \sqrt{n}}{\sqrt{\pi}} \times$$

$$\times \int_0^\infty G_{n+4,\ 3n+2}^{2n,\ 4} \left[\frac{4}{(2n)^{2n}} \cdot \frac{y^{2n}}{p^2} \middle| \begin{matrix} \Delta(2, -\tfrac{1}{2}-(\lambda+m)), \Delta(2, -\tfrac{1}{2}-(\lambda-m)), \Delta(n, -\tfrac{1}{4}-\tfrac{1}{2}v) \\ \Delta(n, \tfrac{1}{4}-\tfrac{1}{2}v), \Delta(n, \tfrac{1}{4}+\tfrac{1}{2}v), \Delta(2, -2\lambda), \Delta(n, -\tfrac{1}{4}-\tfrac{1}{2}v) \end{matrix} \right] \phi(y)\, dy \qquad (4.4)$$

where R $[\lambda \pm m+(\frac{1}{2}\pm v)\ 1/n+3/2 > O$ R $(\frac{1}{2}\pm v)>O$ and $f(x^{1/n})$, $\phi(x)$ both belong to class L (O, ∞)

Further if we put $n=2$ in the above corollary it reduces to the theorem given by Varma C. B. L. (9).

Theorem 4 :—If $f(x^{r/s})$, $\phi(x)$ L (O, ∞)

also R $[v+3/2]>O$, R $[\lambda \pm m+(v+3/2)\, r/s+3/2]>O$

$$\psi(p) \frac{W}{\lambda,k,m} f(x^{r/s})$$

and

$$\phi(p) \overset{Y}{\underset{v}{=}} f(x)$$

then

$$\psi(p) = (2\pi)^{\frac{1}{2}-r} \times (2r)^{\frac{1}{2}} \times (2s)^{\frac{1}{2}} \times$$

$$\times \int_0^\infty G^{s,\, s+4r}_{s+4r,\, 3s+2r}\left[\frac{(2r)^{2r}}{(2s)^{2s}} \cdot \frac{y^{2s}}{p^{2r}} \middle| \begin{matrix} \Delta(s, \frac{3}{4}+\frac{1}{2}v), \Delta(2r, -\frac{1}{2}-(\lambda+m)), \\ \Delta(2r, -\frac{1}{2}\ (\lambda-m)) \\ \Delta(s, \frac{3}{4}+\frac{1}{2}v), \Delta(2r, -2\lambda), \\ \Delta(s, \frac{1}{4}-\frac{1}{2}v), \Delta(s, \frac{1}{4}+\frac{1}{2}v) \end{matrix}\right] \phi(y)\, dy \quad (5\cdot 1)$$

If we put $r=s$ then by virtue of the formulas (2.7) and (2.8) we obtain the theorem that if

$$\psi(p) \frac{W}{\lambda,k,m} f(x)$$

and

$$\phi(p) \overset{Y}{\underset{v}{=}} f(x)$$

then

$$\psi(p) = \frac{2^{3/2}}{\pi} \cdot \frac{1}{p^{v+3/2}} \times \frac{\Gamma\left(2+\frac{\lambda \pm m+v}{2}\right) \Gamma\left(3/2+\frac{\lambda \pm m+v}{2}\right)}{\Gamma(7/4+(v/2+\lambda))\ \Gamma(9/4+(v/2+\lambda))\ \Gamma(v+3/2)} \times$$

$$\times \int_0^\infty {}_5F_4\left[\begin{matrix} 1, 2+\frac{\lambda \pm m+v}{2}, 3/2+\frac{\lambda \pm m+v}{2}, \\ 7/4+v/2+\lambda, 9/4+v/2+\lambda, v+3/2, 3/2, \end{matrix} -\frac{y^2}{p^2}\right] y^{v+3/2} \phi(y)\, dy \qquad (5.2)$$

provided that R $[\lambda \pm m+(v+3/2)+3/2]>O$, R $[v+3/2]>O$ and the functions $f(x)$ and $\phi(x)$ both belong to the class L (O, ∞).

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REFERENCES

1. Erdelyi, A., Tables of Integral Transforms Vol. I.
2. Erdelyi, A-, Higher Transcendedtal functtions Vol. I.
3. Meizer, C. S., *Proc. Amsterdam Akad. Wet.* 43 (1940).
4. Meizer, C. S., *Proc. Nedcral Akad. Wet Amsterda* (1941).
5. Mainra, V. P., *Bull. Cal. Math. Soc.* 53 (1961) p. 23-31.
6. Saxena, R. K., *Proc. of Nat. Akad. of Scien.* Vol. 26 (1960).
7. Titchmarsh, E. C., Introduction to the theory of fourier Integrals (1937).
8. Varma, R. S., *Proc. Nat. Acad. Sci. India* 20A. 209 (51).
9. Varma, C. B., *Proc. Nat. Acad. Sci.* India, 32 (1963).

STUDY OF HETEROPOLYSALTS WITH TETRAVALENT VANADIUM AS THE CENTRAL ATOM

PART I—PRECIPITATION OF VANADYL TUNGSTATE

By

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ABSTRACT

Vanadyl sulphate, when mixed with sodium tungstate results in the formation of a precipitate. A conductometric study of the reaction by monovariation method showed that the precipitate is formed by the reaction of equimolar vanadyl sulphate and sodium tungstate in 1: 2 ratio. The composition has been further confirmed by analysis. The substance can be represented as $Na_2[VO(WO_4)_2]$.

Alkali tungstates are known to be precipitants for several cations.[1] The cationic nature of the vanadyl ion (VO^{++}) is well established, its ferro and ferricyanides[2] being precipitated. In this work an attempt has been made to study the reaction between vanadyl sulphate and sodium tungstate, this being a prelude to the study of heterepolytungstates with tetravalent vanadium as the central atom.

EXPERIMENTAL

Solutions were prepared by dissolving analysed samples of vanadyl sulphate (B. D. H.) and sodium tungstate (B. D. H.) in double distilled water. Conductance was used as the index for precipitation studies.

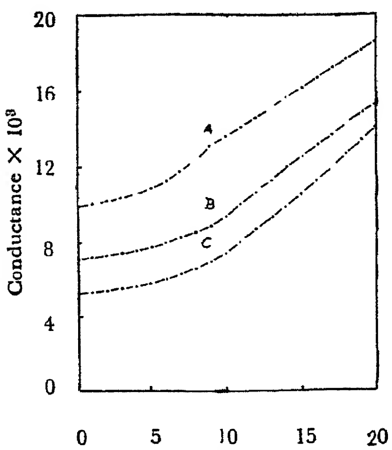

Fig. 1. Vol. of sodium tungstate in ml.

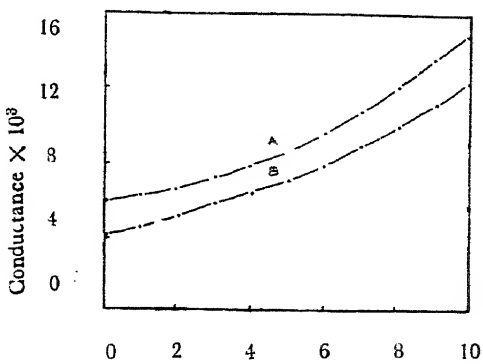

Fig. 2. Vol. of vanadyl sulphate in ml.

The curves A, B, C, in fig. 1 were obtained when varying amounts of sodium tungstate solutions were added to a constant volume (5 ml.) of equimolar vanadyl

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sulphate solution following the monovariation method, the total volume being made 25 ml. in each case by adding distilled water. Curves A, B and C correspond to M/20, M/30 and M/40 equimolar solutions of the reactants, respectively. The same reaction when studied in the reverse way, *i. e.* by varying the volume of equimolar vanadyl sulphate, yielded two curves A and B in fig. 2, standing respectively for M/20 and M/30 equimolar solutions.

The breaks in the curves in fig. 1 and 2 correspond to the formation of a precipitate by the interaction of one molecule of vanadyl sulphate with two molecules of sodium tungstate. The composition of the precipitate will be $Na_2\ VO(WO_4)_2$ according to the following equations :

$$VO\ SO_4 + Na_2WO_4 \rightleftharpoons VO\ WO_4 + Na_2\ SO_4$$

$$VO\ WO_4 + Na_2WO_4 \rightleftharpoons Na_2\ [VO\ (WO_4)_2]$$

The composition of the precipitate has been further established by analytical studies. The precipitate was filtered, washed and dried. Weighed quantities of the precipitate were dissolved in hydrochloric acid and the amounts of tungsten and vanadium present were determined by precipitating barium tungstate[3] and titrating the filtrate against standard potassium permanganate[3], respectively. The results tabulated below correspond to the composition $Na_2[VO(WO_4)_2]$

TABLE I

$Na_2\ [VO\ (WO_4)_2]$ required—V=8.652%, W=62.254%

Found— V=8.633%, W=62.236%.

The compound can also be represented as $Na_2O.VO_2.2WO_3$ and represents the first stage of the condensation of tungsten and tetravalent vanadium anhydrides to form the polysalt.

Thanks are due to Prof. S. M. Mukherji, Head of the Department of Chemistry, Kurukshetra University for providing the laboratory facilities.

REFERENCES

1. C. M. Gupta and R. S. Saxena, *J. Ind. Chem. Soc.* (1958), 35, 830, 36, 845. (1959).

2. P. K. Bhattacharya and S. N. Banerji, *Proc. Natl. Acad. Sci., India*, (1961), 30, 33 and 39.

3. A. I. Vogel, A Text book of Quantitative Inorganic Analysis, Longmans Green and Co., N. Y. 1955, pp. 492, 319.

CATALYTIC ACTIVITY OF KOHLSCHÜTTER'S RED SILVER SOL

STUDY OF THE DECOMPOSITION OF HYDROGEN PEROXIDE

By

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ABSTRACT

Silver sols of different grain sizes have been prepared by Kohlschütter's method. Under controlled conditions a red sol with λ_{max} at 430 mμ and optical density ($\log I_0/I_t$) = 1.380 is obtained. The catalytic decomposition of hydrogen peroxide by the nascent red sol and its aged samples has been investigated. It has been shown that the rate of decomposition of H_2O_2 in the persence of the colloid catalyst is a second order process for dilute solutions of H_2O_2 and the second order changes to first when the concentration of H_2O_2 is increased. The energy of activation for the catalytic decomposition with the nascent sol is 16,290 calories and increases to 17.850 calories for the 17 days aged sample. Adopting colorimetric methods the changes in the stability of the colloid catalyst with age have been investigated. A decrease in stability and catalytic activity with age is observed. A suitable mechanism for the colloid catalysed decomposition is proposed and discussed.

The catalytic decomposition of hydrogen peroxide by platinum, gold and ferric hydroxide sols has been investigated by Bredig[1], Suito[2] and Kepfer and Walton[3]. Extensive study of the decomposition of hydrogen peroxide in aqueous solutions by several hydrous oxides has been made by Ghosh and coworkers[4].

It is well known that dissolved argentous silver is not noticeably catalytic for the decomposition of hydrogen peroxide. In fact, 90 wt % hydrogen peroxide can be saturated with silver nitrate without effect. Similarly, the metal and silver oxide in small proportions are soluble and are thereby rendered non-catalytic. In the presence of a little alkali, however, this tolerance is markedly reduced and the addition of argentous ion, metal or oxide quickly produces a precipitate which initiates the decomposition[5]. We have observed that colloidal silver of different grain sizes considerably catalyse the decomposition of hydrogen peroxide. In a recent communication,[6] we have described the catalytic decomposition of hydrogen peroxide by Kohlschütter's violet silver sol. In this paper we are presenting our results with the red variety which by spectrophotometric studies we have shown to have average bigger particles yet is copable of exhibiting considerable catalytic effects.

EXPERIMENTAL

The red coloured Kohlschütter's silver sol was prepared by the reduction of a freshly precipitated and thoroughly washed sample of Ag_2O by hydrogen under controlled conditions. It was observed that slow reduction of the Ag_2O suspension at a temperature of 50 ± 0.1°C gave a stable sample of the red sol[7].

An aqueous solution of Merckozone was used throughout the investigations which contained some stabiliser. No attempt was made to purify it so that the rate of self decomposition could be kept low.

For kinetic study several 100 ml. pyrex conical flasks each containing 25 ml. of Kohlschütter red sol were thermostated at 30 ± 0.1°C. Another flask con-

taining a standardised hydrogen peroxide solution was thermostated at the same temperature. The flasks were wrapped in black opaque cloth to avoid light influences. 5 ml. of hydrogen peroxide solution was then pipetted out and added to each flask after an interval of one minute. The flasks were taken out one after the other after fixed intervals of times and the reaction was checked by adding 20 ml. of ice cold sulphuric acid (4N). The undecomposed H_2O_2 was titrated against a standard solution of $KMnO_4$ from a microburette.

In the following tables the results on the rate of decomposition of H_2O_2 by fresh red silver sol are shown. The silver content of the sol was .001038 gm/ 25 ml. of the sol with pH=9.15.

The decomposition was studied at four different concentrations at 30±0.1°C. k_1 and k_2 represent first order and second order constants respectively.

Fresh Sol.

TABLE 1

Rate of decomposition of

Time Minutes	0.2N H_2O_2 $k_1/2.303$ $\times 10^4$	0.1N H_2O_2 $k_1/2.303$ $\times 10^4$
0	...	...
10	9.70	67.50
20	9.75	59.60
30	9.56	58.83
40	9.52	54.75
50	9.34	51.30
60	...	51.30

TABLE 2

Rate of decomposition of

	0.05 N H_2O_2		0.025 N H_2O_2	
Time Minutes	$k_1/2.303$ $\times 10^3$	k_2 $\times 10^4$	$k_1/2.303$ $\times 10^3$	k_2 $\times 10^3$
0	...	...	...	...
5	21.36	55.75	23.84	6.31
10	19.11	55.27	30.28	10.08
15	17.32	54.54	26.52	10.00
20	18.84	69.04	23.42	9.70
25	17.08	66.73	21.50	9.79
30	16.31	69.54	20.66	10.55

Energy of activation

The catalytic decomposition of 0.1 N H_2O_2 was also investigated at temperatures 25° and 35°C. and the energy of activation calculated from the mean values of the first order rate constants comes out to be 16,290 calories for the temperature range 25°—35°C.

10 days aged sol

The red sol was allowed to age for 10 days and the investigations made earlier with the fresh sol were repeated with the aged sample. The pH of the sol was 9.15.

TABLE 3

Rate of decomposition of

Time Minutes	0.2 N H_2O_2 $k_1/2.303 \times 10^4$	0.1 N H_2O_2 $k_1 2.303 \times 10^4$
0	...	...
10	15.90	49.60
20	15.75	47.35
30	14.60	43.20
40	14.37	42.20
50	13.50	40.14
60	...	44.33

TABLE 4

Rate of decomposition of

	0.05 N H_2O_2		0.025 N H_2O_2	
Time Minutes	$k_1/2.303 \times 10^3$	$k_2 \times 10^4$	$k_1/2.303 \times 10^3$	$k_2 \times 10^4$
0	...	...	...	...
5	14.74	39.96	15.14	39.04
10	14.51	39.60	14.15	38.54
15	13.56	39.83	13.38	39.15
20	14.96	49.60	13.38	42.59
25	13.99	49.69	11.90	39.36
30	12.56	46.03	12.56	46.03

Energy of activation :

The energy of activation of the reaction with the 10 days aged sol was found to be 17,560 calories for the temperature range 25°—35°C,

17 days aged sol :

The red sol was allowed to age for 17 days and the investigations made earlier with the fresh sol were repeated with the aged sample. The pH of the sol was 8.90.

TABLE 5

Rate of decomposition of

Time Minutes	0.2 N H_2O_2 $k_1/2.303 \times 10^4$	0.1 N H_2O_2 $k_1/2.303 \times 10^4$
0	...	...
10	11.40	29.20
20	11.60	29.25
30	11.43	29.80
40	11.20	29.50
50	10.40	29.05
60	10.75	28.98

TABLE 6

Rate of decomposition of

Time Minutes	0.05 N H_2O_2 $k_1/2.303 \times 10^4$	Time Minutes	0.025 N H_2O_2 $k_1/2.303 \times 10^4$
0	...	0	...
10	55.50	5	30.00
20	57.30	10	38.10
30	57.53	15	40.45
40	56.90	20	34.86
50	57.80	25	38.10
60	55.26	30	30.08

Energy of activation :

The energy of activation of the reaction with the 17 days aged sol was obtained as with the fresh sol and it is 17,850 calories for the temperature range 25° – 35°C.

Maximum absorbance and optical density

Employing a Unicam S P. 500 spectrophotometer the maximum absorbance of the nascent sol is found to be at 430 mμ and this value remains unchanged as the ageing progresses. The changes in the optical densities with age corresponding to this wavelength are recorded in the following table :

TABLE 7

Sol	Optical density (log I_0/I_t)
Fresh	1.360
10 days aged	1.125
17 days aged	0.960

Stability of the sol

Our results on the investigations of the change of the stability of the sol with age employing a Klett-Summerson photoelectric colorimeter are summarised in the following table :

TABLE 8

Sol	Amount of electrolyte necessary to bring Klett colorimeter reading to 172 in 30 minutes.	
	M/10 KNO_3	M/200 $Ba(NO_3)_2$
Fresh	1.00 ml	0.84 ml
10 days aged	0.86 ml	0.80 ml
17 days aged	0.60 ml	0.70 ml

DISCUSSION

From the foregoing tables the following conclusions are drawn.

1. The rate of decomposition of H_2O_2 in the presence of colloid catalyst is a second order process for dilute solutions of H_2O_2 and the second order changes to first order when the concentration of H_2O_2 is increased. Percentage decomposition of H_2O_2 decreases with the increasing concentration of H_2O_2.

evident from figs. 1-3 that a decrease in concentration of H_2O_2 in the system produces a departure from the straight lines with a downward bending when log $a/(a\text{-}x)$ is plotted against t. Such a bending curve indicates a shift in the first order reaction towards second order (figs. 1c, 1D, 2c). Figs. 1A,B' and 2A,B and 3A,B are straight lines showing that reaction is a first order process for 0.2 N H_2O_2 and 0.1 N H_2O_2 throughout the course of investigation. Ageing decreases the stability of the sol, hence addition of 0.025 NH_2O_2 immediately turned the colour of the red sol to violet indicating an increase in the size of the colloidal particles. Such an increase resulted in a decrease in the rate constants for the decomposition of 0.025 N H_2O_2 instead of an increase as compared to rate cons-

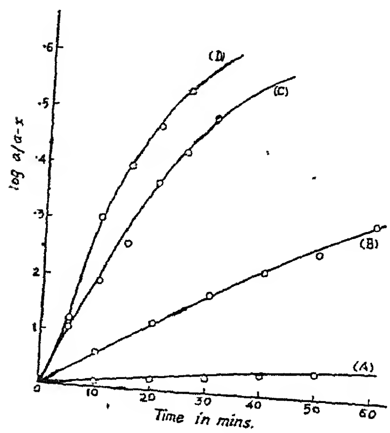

Fig. 1. Plot of time *versus* log $a/(a-x)$ for red sol.
Conc. of H_2O_2-A : N/5 ; B : N/10 ;
C : N/20 ; D : N/40

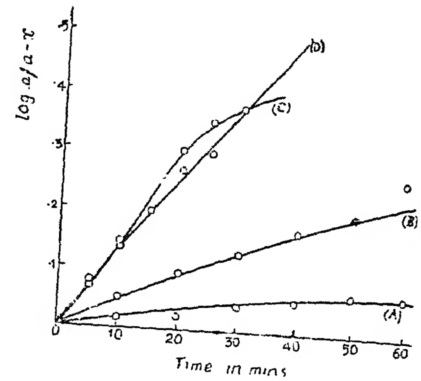

Fig. 2. Plot of time *versus* log $a/(a-x)$ for to days' aged Sol.
Conc. of H_2O_2 same as in Fig. 1

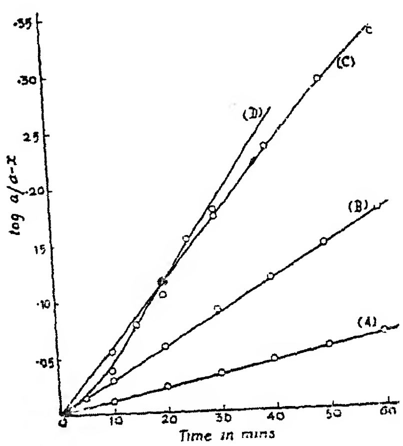

Fig. 3. Plot of time *versus* log $a/(a\text{-}x)$ for 17 days' aged Sol.
Conc. of H_2O_2 same as in Fig 1

tants for 0.05 N H_2O_2 (table 6). The increase in the particle size brought about a shift in the order of the decomposition which is now first order and not second order as with the fresh sol.

2. The rate of decomposition has decreased slightly with the 10 days aged sol and appreciably with the 17 days aged sol.

3. Employing the nascent sol, it is observed that the activation energy is 16,290 calories which increases to 17,560 calories with the 10 days aged sol and ultimately attains the value of 17,850 calories for the 17 days aged sol. This clearly indicates that as the ageing progresses the rate of decomposition decreases and the catalytic activity of the sol diminishes.

Absorbance measurements show that λ_{max} for the red sol is at 430 mμ. Comparative values of absorbances at this λ_{max}, however, indicate that the values decrease with age. This shows that the overall absorbing surface is decreased and the particles become bigger and bigger as ageing continues.

Decomposition Mechanism :

The decomposition of hydrogen peroxide in the presence of different catalysts gives usually the same simple products viz., water and oxygen according to the following :—

$$2H_2O_2 \longrightarrow 2H_2O + O_2$$

The pathways suggested for such a decomposition, however, may not be the same in each case. The mechanism of homogeneous gas phase decomposition of hydrogen peroxide has been given from energy considerations[8]. The rupture of hydrogen peroxide molecule can take place in only two possible ways viz.,

(*a*) The breaking of O-O bond to form OH radicals.

(*b*) The breaking of O-H bond to give an HO_2 radical and a H atom.

It has been reported that the energy required to break a O-O bond is 52 K cal. per mole while the energy required to break a O-H bond is 90 K Cal. per mole. Therefore, breaking of O-O bond is more probable in the first step of decomposition of hydrogen peroxide producing OH radicals which initiate a chain mechanism. The above hypothesis is supported by the work of Urey, Dawsey and Rice[9].

It is well known that adsorption of the reactants on the surface of a solid catalyst is the primary process in all heterogeneous catalysed reactions. It appears, therefore, that when H_2O_2 is added to the colloidal silver an adsorption complex is first formed on the colloidal surface and H_2O_2 becomes reactive due to weakening of the bonds. Thus

$$\underset{\text{(Colloid surface)}}{[S]} + H_2O_2 \longrightarrow \underset{\text{(Adsorption complex)}}{[S]\,OH-OH} \qquad \ldots \qquad \ldots \quad (1)$$

This adsorption complex reacts with a molecule of the hydrogen peroxide as follows :—

$$[S]\,OH-OH + H_2O_2 \longrightarrow [S]\,OH + HO_2 + H_2O \qquad \ldots \quad (2)$$

The HO_2 radical produced above initiates the chain reaction

$$HO_2+H_2O_2 \longrightarrow H_2O+O_2+OH \quad \cdots \quad (3)$$

$$OH+H_2O_2 \longrightarrow H_2O+HO_2 \quad \cdots \quad (4)$$

To explain our results the step (2) appears to be the rate determining step. The concentration of the adsorption complex is proportional to the concentration of the hydrogen peroxide when dilute solutions of H_2O_2 are employed hence the step (2) will be a second order process in dilute solutions of H_2O_2. At higher concentrations of H_2O_2, the concentration of the adsorption complex becomes independent of H_2O_2 concentration and the reaction becomes a first order process. When the surface of the catalyst, however, is very reactive the weakening of the bonds take place rapidly and the rate determining step becomes fast so much so that the decomposition process becomes zero order process. In the case of yellow sol, the surface being very reactive the order of the decomposition changes from second order to first order and tends to become zero order when concentration of H_2O_2 is increased as expected from the above mechanism. The surface of the red sol being less reactive the order of decomposition changes from second order to first order as the concentration of H_2O_2 is increased because the step (2) does not become considerably fast. The above mechanism, therefore, clearly explains all the results of our investigations presented in this paper.

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REFERENCES

1. Bredig, G., and von Berneck, R. M., *Z. Physik. Chem.*, 31, 258 (1899); 37, 323 (1901).
2. Tamura, K., Tarami, K., and Suito, E., *Nippon Kagaku Zasshi*, 80, 10 (1959).
3. Kepfer, R. J., and Walton, J. H., *J. Phys. Chem.*, 34, 557 (1931).
4. Tiwari, S. N., and Ghosh, S., *Kolloid-Z.*, 124, 69 (1951).
   Kulshrestha, N. K., Dey, A. K., and Ghosh, S., *Kolloid-Z.*, 132, 143 (1953).
   Gupta, S. R., and Ghosh, S., *Kolloid-Z.*, 135, 143 (1954).
   Beg, M. A., and Ghosh, S., *Bull Natl. Inst. Sci., India*, 2, 297 (1959).
   Jalaj, K., and Ghosh, S., *Proc. Natl. Acad. Sci. India*, 32 A, 171 (1962).
5. Von Euler, H., and Jansson, B., *Monatsh*, 53-54, 1014 (1929).
6. Singh, V., and Mushran, S. P., *Ind. J. Chem.*, (1965).
7. Singh, V., Doctoral Thesis, Allahabad, (1964).
8. Schumb, W. C., Satterfield, C. N., and Wentworth, R. L., Hydrogen Peroxide, 456-467 (1955).
9. Urey, H. C., Dawsey, L. H. and Rice, F. O., *J. Am. Chem. Soc.* 51, 1371 (1929).

ON AN ASSOCIATED FUNCTION OF ULTRASPHERICAL POLYNOMIALS

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1. The Ultraspherical polynomials $P_n^{\lambda}(x)$ are given by

$$(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{\lambda}(x)\, t^n$$

and let

$$(1.1) \qquad f_n^{\lambda}(x) = \frac{n!}{(\lambda)_n} P_n^{\lambda}(x)$$

The pure recurrence relation of the Ultraspherical polynomials is given by [1, p. 175]

$(u+1)\ P_{n+1}^{\lambda}(x) = 2\ (n+\lambda)\ x\ P_n^{\lambda}(x) - (n+2\lambda-1)\ P_{n-1}^{\lambda}(x)$ which for the function $f_n^{\lambda}(x)$ of (1.1) becomes

$$(12.) \qquad f_{n+1}^{\lambda}(x) = 2x f_n^{\lambda}(x) - a_{n-1} f_{n-1}^{\lambda}(x)$$

where a_{n-1} of (1.2) is given by the relation

$$(1.3) \qquad a_n = \frac{(n+1)\,(n+2\lambda)}{(n+1+\lambda)\,(n+\lambda)}$$

From (1.3) we easily find

$$(1.4) \qquad a_n - a_{n-r} = \frac{r\lambda\,(\lambda-1)\,(2n+2\lambda-r+1)}{(n+\lambda-r)_2\,(n+\lambda)_2}$$

we have

$$P_0^{\lambda}(x) = 1,\ P_1^{\lambda}(x) = 2\lambda x,\ P_2^{\lambda}(x) = 2\lambda\,(1+\lambda)x^2 - \lambda$$

$$P_3^{\lambda}(x) = \frac{4\lambda}{3}(1+\lambda)\,(2+\lambda)\,x^3 - 2\lambda x\,(1+\lambda)$$

and from them we easily deduce

$$f_0^{\lambda}(x)\equiv 1,\ f_1^{\lambda}(x)=2x,\ f_2^{\lambda}(x)=4x^2-\frac{2}{1+\lambda}$$

$$f_3^{\lambda}(x)=8x^3-\frac{12x}{\lambda+2} \tag{1.5}$$

According to the Associated function of the Hermite Polynomials [2, p. 116 121], the Associated functions of the Ultraspherical polynomials are defined as

$$\Delta_{n,h,k,x}(z)=z_{n+h}(x)\,z_{n+k}(x)-z_n(x)\,z_{n+h+k}(x)\quad k\geqslant h \tag{1.6}$$

where $z_n(x)$ is either $f_n^{\lambda}(x)$, or $P_n^{\lambda}(x)$.

In this paper expansions and recurrence relations of the functions given by (1.6) have been derived with the help of (1.2). The values given by (1.4) and (1.5) are extensively used.

2. Now

$$\Delta_{n,1,1:x}(f)=[f_{n+1}^{\lambda}(x)]^2-f_n^{\lambda}(x)\,f_{n+2}^{\lambda}(x)$$

$$=f_{n+1}^{\lambda}(x)\,[2x f_n^{\lambda}(x)-a_{n-1}f_{n-1}^{\lambda}(x)]$$

$$-f_n^{\lambda}(x)\,[2x f_{n+1}^{\lambda}(x)-a_n f_n^{\lambda}(x)]$$

$$=a_n[f_n^{\lambda}(x)]^2-a_{n-1}f_{n-1}^{\lambda}(x)\,f_{n+1}^{\lambda}(x) \tag{2.1}$$

$$=a_{n-1}\,\Delta_{n-1,1,1:x}(f)+(a_n-a_{n-1})\,[f_n^{\lambda}(x)]^2 \tag{2.2}$$

Repeated use of (2.2) with the help of (1.4) and (1.5) give

$$\Delta_{n,1,1:x}(f)=2\lambda(\lambda-1)\sum_{r=0}^{n}\frac{(-n)_r\,(-n-2\lambda+1)_r}{(-n-\lambda)_r\,(-n-\lambda+1)_r\,(n+\lambda-1-r)_3}\times\left\{f_{n-r}^{\lambda}(x)\right\}^2 \tag{2.3}$$

Writing (2.1) as

$$\Delta_{n,1,1:x}(f)=a_n\,\Delta_{n-1,1,1:x}(f)+(a_n-a_{n-1})\,f_{n-1}^{\lambda}(x)\,f_{n+1}^{\lambda}(x) \tag{2.4}$$

and on the same lines (2.4), (1.4) and (1.5) give

$$(2.5)\qquad \Delta_{n,1,1:x}(f) = \frac{2\,(n+1)!\,(1+2\lambda)_n}{[(1+\lambda)_n]^2\,(n+1+\lambda)} + 2\lambda\,(\lambda-1)$$

$$\times \sum_{r=0}^{n-1} \frac{(-n-1)_r\,(-n-2\lambda)_r}{(-n-\lambda)_r\,(-n-\lambda-1)_r}\;\frac{f_{n-r-1}^{\lambda}(x)\,f_{n-r+1}^{\lambda}(x)}{(n-r-1+\lambda)}$$

Now

$$(2.6)\qquad \Delta_{n,1,1:x}(P) = \left\{ P_{n+1}^{\lambda}(x) \right\}^2 - P_n^{\lambda}(x)\,P_{n+1}^{\lambda}(x)$$

and using (1.1), (2.6) reduces to

$$(2.7)\qquad \Delta_{n,1,1:x}(P) = \frac{-(1-\lambda)}{(n+1)\,(\lambda+n+1)} P_n^{\lambda}(x)\;P_{n+2}^{\lambda}(x) + \left[\frac{(\lambda)_{n+1}}{(n+1)!}\right]^2 \Delta_{n,1,1:x}(f)$$

$$(2.8)\qquad = \frac{-(\lambda-1)}{(\lambda+n)\,(n+2)} \left[P_{n+1}^{\lambda}(x)\right]^2 + \frac{(\lambda)_n\,(\lambda)_{n+2}}{n!\,(n+2)!} \Delta_{n,1,1:x}(f)$$

Now (2.7) with the help of (2.3) and (1.1), reduces to

$$\Delta_{n,1,1:x}(P) = \frac{(1-\lambda)}{(n+1)\,(\lambda+n+1)}\; P_n^{\lambda}(x)\,P_{n+2}^{\lambda}(x) + \frac{2\lambda\,(\lambda-1)\,(\lambda+n)^2}{(n+1)^2}$$

$$\times \sum_{r=0}^{n} \frac{(-n-2\lambda+1)_r\,(-\lambda-n+1)_r}{(-n)_r\,(-\lambda-n)_r\,(n+\lambda-1-r)_3} \left[P_{n-r}^{\lambda}(x) \right]^2$$

and with the help of (2.5) and (1.1), it becomes

$$\Delta_{n,1,1:x}(P) = -\frac{(1-\lambda)}{(n+1)\,(\lambda+n+1)} P_n^{\lambda}(x)\,P_{n+2}^{\lambda}(n) + \frac{\lambda(2\lambda)_{n+1}}{(n+1)!\,(1+\lambda+n)}$$

$$+ \frac{2\lambda(\lambda-1)\,(\lambda+n-1)}{(n)_2} \sum_{r=0}^{n-1} \frac{(-n-2\lambda)_r\,(-\lambda-n+2)_r}{(-n-\lambda-1)_r\,(-n+1)_r\,(n-r-1+\lambda)_3}$$

$$P_{n-r-1}^{\lambda}(x)\;P_{n-r+1}^{\lambda}(x)$$

Similarly (2.8) with the help of (2.3) and (2.5) becomes respectively

$$\Delta_{n,1,1:x}(P) = -\frac{(\lambda-1)}{(\lambda+n)\,(n+2)} \left[P_{n+1}^{\lambda}(x) \right]^2 + \frac{2(\lambda+n)_2\,\lambda\,(\lambda-1)}{(n+1)_2}$$

$$\times \sum_{r=0}^{n} \frac{(-n-2\lambda+1)_r\,(-\lambda-n+1)_r}{(-n)_r\,(-\lambda-n)_r\,(n+\lambda-1-r)_3} \left[P_{n-r}^{\lambda}(x) \right]^2$$

and

$$\Delta_{n,1,1:x}(P) = -\frac{(\lambda-1)}{(\lambda+n)(n+2)}\left[P_{n+1}^{\lambda}(x)\right]^2 + \frac{2\lambda^2(1+2\lambda)_n}{n!(\lambda+n)(n+2)}$$

$$+\frac{2\lambda(\lambda-1)(\lambda+n-1)(\lambda+n+1)}{n(n+2)}$$

$$\times \Sigma_{r=0}^{n-1}\frac{(-n-2\lambda)_r(-\lambda-n+2)_r P_{n-r-1}^{\lambda}(x) P_{n-r+1}^{\lambda}(x)}{(-n-\lambda-1)_r(-n+1)_r(n-r-1+\lambda)_3}$$

we have from (1.6)

$$\Delta_{n,1,2:x}(f) = f_{n+1}^{\lambda}(x) f_{n+2}^{\lambda}(x) - f_n^{\lambda}(x) f_{n+3}^{\lambda}(x)$$

$$= 2x\Delta_{n,1,1:x}(f) + \frac{2\lambda(\lambda-1)}{(n+\lambda)_3} f_n^{\lambda}(x) f_{n+1}^{\lambda}(x)$$

which can also be written as

$$\Delta_{n,1,2:x}(f) = a_{n+1} f_n^{\lambda}(x) f_{n+1}^{\lambda}(x) - a_{n-1} f_{n-1}^{\lambda}(x) f_{n+2}^{\lambda}(x)$$

(2.9) $$\equiv a_{n-1}\Delta_{n-1,1,2:x}(f) + (a_{n+1} - a_{n-1}) f_n^{\lambda}(x) f_{n+1}^{\lambda}(x)$$

From (2.9) we can get the expansion of $\Delta_{n,1,2:x}(f)$ simlar to (2.3) and (2.5).

Now

$$\Delta_{n,1,3:x}(f) = 2x\,\Delta_{n,1,2:x}(f) + a_{n+2} f_n^{\lambda}(x) f_{n+2}^{\lambda}(x) - a_{n+1}\left[f_{n+1}^{\lambda}(x)\right]^2$$

$$= 2x\,\Delta_{n,1,2:x}(f) - \frac{(n+2)(n+1+2\lambda)}{(n+2+\lambda)(n+1+\lambda)}\Delta_{n,1,1:x}(f)$$

$$+\frac{2\lambda(\lambda-1)}{(n+\lambda+1)_3} f_n^{\lambda}(x) f_{n+2}^{\lambda}(x)$$

and also

$$\Delta_{n,1,3:x}(f) = 2x\,\Delta_{n,1,2:x}(f) - a_{r+2}\Delta_{n,1,1:x}(f) + (a_{n+2} - a_{n+1})\left[f_{n+1}^{\lambda}(x)\right]^2$$

by which we can have similer expansions of $\Delta_{n,1,3:x}(f)$

3. We have from (1.6)

$$\Delta_{n,1,k:x}(f) = f_{n+1}^{\lambda}(x) f_{n+k}^{\lambda}(x) - f_n^{\lambda}(x) f_{n+k+1}^{\lambda}(x)$$

$$= a_{n+k-1} f_n^{\lambda}(x) f_{n+k-1}^{\lambda}(x) - a_{n-1} f_{n-1}^{\lambda}(x) f_{n+k}^{\lambda}(x)$$

$$(3.1) \qquad = a_{n-1}\,\Delta_{n-1,1,k:x}(f) + (a_{n+k-1} - a_{n-1})\, f_n^{\lambda}(x)\, f_{n+k-1}^{\lambda}(x)$$

Repeated use of (3.1) gives

$$(3.2) \qquad \Delta_{n,1,k:x}(f) = k\lambda(\lambda - 1)$$

$$\times \sum_{r=0}^{n} \frac{(-n)_r\,(-n-2\lambda+1)_r\,(2n+k+2\lambda-1-2r)}{(-n-\lambda)_r\,(-n-\lambda+1)_r\,(n+\lambda-1-r)_2\,(n+\lambda+k-1-r)_2}$$

$$+ f_{n-r}^{\lambda}(x)\, f_{n+k-r-1}^{\lambda}(x)$$

of which (2.3) is a particular case and also

$$(3.3) \qquad \Delta_{n,1,k:x}(f) = a_{n+k-1}\,\Delta_{n-1,1,k:x}(f) + (a_{n+k-1} - a_{n-1})$$

$$f_{n-1}^{\lambda}(x)\, f_{n+k}^{\lambda}(x)$$

Repesated use of (3.3) gives

$$(3.4) \qquad \Delta_{n,1,k:x}(f) = \frac{(k)_{n+1}\,(k+2\lambda-1)_{n+1}}{(k+\lambda)_{n+1}\,(k+\lambda-1)_{n+1}}\, f_{k-1}^{\lambda}(x) + k\lambda(\lambda-1)$$

$$\times \sum_{r=0}^{n-1} \frac{(-n-k)_r\,(-n-k+1-2\lambda)_r\,(2n+k-2r+2\lambda-1)}{(-n-k+1-\lambda)_r\,(-n-k-\lambda)_r\,(n-r-1+\lambda)_2\,(n+k-r-1+\lambda)_2}$$

$$f_{n-r-1}^{\lambda}(x)\, f_{n+k-r}^{\lambda}(x)$$

of which (2.5) is a particular case.

We also have

$$\Delta_{n,1,k:x}(f) = 2x\,\Delta_{n,1,k-1:x}(f) + a_{n+k-1} f_n^{\lambda}(x)\, f_{n+k-1}^{\lambda}(x)$$

$$- a_{n+k-2} f_{n+1}^{\lambda}(x)\, f_{n+k-2}^{\lambda}(x)$$

Now

$$\Delta_{n,1,k:x}(P) = P_{n+1}^{\lambda}(x)\, P_{n+k}^{\lambda}(x) - P_n^{\lambda}(x)\, P_{n+k+1}^{\lambda}(x)$$

and using (1.1) we get

$$(3.5)\qquad \triangle_{n,1,k:x}(P) = -\frac{k(1-\lambda)}{(n+1)(\lambda+n+k)} P_n^{\lambda}(x)\, P_{n+k+1}^{\lambda}(x) + \frac{(\lambda)_{n+k}(\lambda)_{n+1}}{(n+k)!\,(n+1)!} \triangle_{n,1,k:x}(f)$$

$$(3.6)\qquad = \frac{-k(\lambda-1)}{(\lambda+n)(n+k+1)} P_{n+1}^{\lambda}(x)\, P_{n+k}^{\lambda}(x) + \frac{(\lambda)_n (\lambda)_{n+k+1}}{n!\,(n+k+1)!} \triangle_{n,1,k:x}(f)$$

From (1.1), (3.2), (3.4), (3.5) and (3.6) we get

$$\triangle_{n,1,k:x}(P) = \frac{-k(1-\lambda)}{(n+1)(\lambda+n+k)} P_n^{\lambda}(x)\, P_{n+k+1}^{\lambda}(x) + \frac{k\lambda(\lambda-1)(\lambda+n)(\lambda+n+k-1)}{(n+1)(n+k)}$$

$$\times \sum_{r=0}^{n} \frac{(-n-2\lambda+1)_r (-\lambda-n-k+2)_r (2n+k+2\lambda-1-2r)}{(-n-\lambda)_r (-n-k+1)_r (n+\lambda-1-r)_2 (n+\lambda+k-1-r)_2} P_{n-r}^{\lambda}(x)\, P_{n+k-r-1}^{\lambda}(x)$$

$$\triangle_{n,1,k:x}(P) = \frac{-k(\lambda-1)}{(\lambda+n)(n+k+1)} P_{n+1}^{\lambda}(x)\, P_{n+k}^{\lambda}(x) + \frac{k\lambda(\lambda-1)(\lambda+n+k-1)_2}{(n+k)_2}$$

$$\times \sum_{r=0}^{n} \frac{(-n-2\lambda+1)_r (-\lambda-n-k+2)_r (2n+k+2\lambda-1-2r)}{(-n-\lambda)_r (-n-k+1)_r (n+\lambda-1-r)_2 (n+\lambda+k-1-r)_2} P_{n-r}^{\lambda}(x)\, P_{n+k-r-1}^{\lambda}(x)$$

$$\triangle_{n,1:k:x}(P) = \frac{-k(1-\lambda)}{(n+1)(\lambda+n+k)} P_n^{\lambda}(x)\, P_{n+k+1}^{\lambda}(x) + \frac{(\lambda)_{n+k}(\lambda)_{n+1}(k)_{n+1}(k+2\lambda-1)_{n+1}(k-1)!\, P_{k-1}^{\lambda}(x)}{(n+k)!\,(n+1)!\,(k+\lambda)_{n+1}(k+\lambda-1)_{n+1}(\lambda)_{k-1}}$$

$$+ \frac{k\lambda(\lambda-1)(\lambda+n-1)_2}{(n)_2} \sum_{r=0}^{n-1} \frac{(-\lambda-n+2)_r (-n-k+1-2\lambda)_r (2n+k-2r+2\lambda-1)}{(-n-k-\lambda)_r (-n+1)_r (n-r-1+\lambda)_2 (n+k-r-1+\lambda)_2} P_{n-r-1}^{\lambda}(x)\, P_{n+k-r}^{\lambda}(x)$$

and

$$\Delta_{n,1,k:x}(P)=\frac{-k(\lambda-1)}{(\lambda+n)(n+k+1)}\,P_{n+1}^{\lambda}(x)\,P_{n+k}^{\lambda}(x)$$

$$+\frac{(\lambda_n)(\lambda)_{n+k+1}(k)_{n+1}(k+2\lambda-1)_{n+1}(k-1)!}{n!\,(n+k+1)!\,(k+\lambda)_{n+1}(k+\lambda-1)_{n+1}(\lambda)_{k-1}}\,P_{k-1}^{\lambda}(x)$$

$$+\frac{(\lambda+n-1)(\lambda+n+k)\,(k\lambda)(\lambda-1)}{n\,(n+k+1)}$$

$$\Sigma_{r=0}^{n-1}\frac{(-\lambda-n+2)_r\,(-n-k+1-2\lambda)_r\,(2n+k-2r+2\lambda-1)}{(-n-k-\lambda)_r\,(-n+1)_r\,(n-r-1+\lambda)_2\,(n+k-r-1+\lambda)_2}\,P_{n-r-1}^{\lambda}(x)\,P_{n+k-r}^{\lambda}(x)$$

Now

$$\Delta_{n,2,k:x}(f)=2x\,\Delta_{n,1,k:x}(f)+(a_{n+k}-a_n)\,f_n^{\lambda}(x)\,f_{n+k}^{\lambda}(x).$$

and using the expansions of $\Delta_{n,1,k:x}(f)$ we can get the expansions of $\Delta_{n,2,k:x}(f)$ we also have

(3.7) $$\Delta_{n,h,k:x}(f)=2x\,\Delta_{n,h,k-1:x}(f)+a_{n+k-1}\,f_n^{\lambda}(x)\,f_{n+k+h-2}^{\lambda}(x)$$

$$-a_{n+k-2}\,f_{n+h}^{\lambda}(x)\,f_{n+k-2}^{\lambda}(x).$$

4. *Particular cases*: The Tchebichef polymials are Ultraspherical polynomials with $\lambda=0$ and 1. From (1.1) it could be seen that

$$T_n(x)=\tfrac{1}{2}f_n^{0}(x) \qquad n=1,2,\ldots\ldots\ldots$$

$$U_n(x)=f_n^{1}(x) \qquad n=0,1,2,\ldots\ldots$$

and for both of them, from (1.3) we get

(4.1) $$a_n\equiv 1$$

giving us the recurrence relation

(4.2) $$z_{n+1}(x)=2x\,z_n(x)-z_{n-1}(x)$$

where $z_n(x)$ is either $T_n(x)$ or $U_n(x)$. The identity (4·1) simplifies all the previous results. Writing

$$\Delta_{n,h,k:x}(z)=z_{n-h}(x)\,z_{n+k}(x)-z_n(x)\,z_{n+k+h}(x)$$

where $z_n(x)$ is either $T_n(x)$ or $U_n(x)$, (3.7) becomes :

$$\triangle_{n,h,k;x}(z) = 2x\,\triangle_{n,h,k-1;x}(z) - \triangle_{n,h,k-2;x}(z)$$

which is a recurrence relation of the type (4.2) and with its help we deduce

$$\triangle_{n,h,k;x}(U) = U_{h-1}(x).\ U_{k-1}(x)$$

$$\triangle_{n,h,k;x}(T) = T_k(x)\,T_n(x) - T_{k+h}(x)$$

The last two results are Trignometic identities.

I am thankful to Dr. B. R. Bhonsle for his guidance in the preparation of this paper.

REFERENCES

1. A. Erde'lyi etc: Higher Transcendental functions Vol II (1953).
2. Chatterjea S. K: On an associated function of Hermite Polynomial, *M. Z.*, 78 (1962) p 116-121.

ASSOCIATED LEGENDRE FUNCTIONS

By

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ABSTRACT

Rangarajan has established the following series :

$$\left(\frac{sm\beta}{sm\alpha}\right)^{m+n} P^{m}_{m+n}(\cos\alpha)=\sum_{r=o}^{n}\binom{2m+n}{r}\left\{\frac{sm\,(\beta-\alpha)}{sm\alpha}\right\}^{r} P^{m}_{m+n-r}(\cos\beta).$$

Using the above series we have evaluated certain integrals

1. INTRODUCTION

Rainville [12] established the following series for Legendre polynomials:

$$\text{(1.1)}\qquad P_n(\cos\alpha)=\left(\frac{sm\alpha}{sm\beta}\right)^{n}\sum_{k=o}^{n}\binom{n}{k}\left\{\frac{sm\,(\beta-\alpha)}{sm\alpha}\right\}^{n-k}.P_k(\cos\beta).$$

Using the above series, Bhonsle [2,3] evaluated certain integrals. Later, Carlitz [7] established a series similar to (1.1) for ultraspherical polynomials. Chattrjea [8,9] made use of the series established by Carlitz to evaluated certain integrals. Recently Yadao [16] has established a series similar to (1.1) for associated Legendre functions. There was a slight omission in the series establisbed by Yadao, which has been rectified by Rangarajan [13]. In this paper we shall make use of the series given by Rangarajan to evaluate certain integrals.

2. SERIES FOR ASSOCIATED LEGENDRE FUNCTIONS

Rangarajan has given the relation

$$\text{(2.1)}\qquad \left(\frac{sm\beta}{sm\alpha}\right)^{m+n} P^{m}_{m+n}(\cos\alpha)=\sum_{r=o}^{n}\binom{2m+n}{r}\left\{\frac{sm\,(\beta-\alpha)}{sm\alpha}\right\}^{r} P^{m}_{m+n-r}(\cos\beta).$$

Putting $\beta=2\alpha$ and $\cos 2\alpha=x$, we obtain

$$\text{(2.2)}\qquad 2^{(m+n)/2}(1+x)^{(m+n)/2}\,P^{m}_{m+n}\left(\sqrt{\frac{1+x}{2}}\right)=\sum_{r=o}^{n}\binom{2m+n}{n-r}P^{m}_{m+r}(x).$$

3. SOME INTEGRALS

Using the orthogonal property of associated Legendre functions

$$(3.1)\qquad \int_{-1}^{1} \left[P_n^m(x)\right]^2 dx = \frac{2}{(2n+1)}\frac{(n+m)!}{(n-m)!},$$

we obtain from (2.2)

$$(3.2)\qquad \int_{-1}^{1} (1+x)^{(m+n)/2}\, P^m_{m+n}\left(\sqrt{\frac{1+x}{2}}\right) P^m_{m+r}(x)\, dx$$

$$= \frac{(2m+n)!}{2^{(m+n-2)/2}\,(2m+2r+1)\,(n-r)!\,r!}\,.$$

Bhonsle and Varma [5] have given the relation

$$(3.3)\qquad \frac{(1-x^2)^{m/2}\,(1+x)^{s-m}}{(s-m)!} = \sum_{r=m}^{s} a_r\, P^m_r(x),$$

where

$$a_r = \frac{2^s\,(2r+1)}{(s+r+1)!\,(s-r)!}$$

Combining this relation with (2.2) and using the orthogonal property, we have

$$(3.4)\qquad \int_{-1}^{1} (1-x)^{m/2}\,(1+x)^{s+n/2}\, P^n_{m+n}\left(\sqrt{\frac{1+x}{2}}\right) dx$$

$$= \frac{(2m+n)!\,(2s+n+1)!}{2^{(m+n-2s-2)/2}\,(s+n+m+1)!\,(2s+1)!}\,.$$

Bloch [6] has recently given the relation

$$(3.5)\qquad e^{tx}\, I_m\left[t(x^2-1)^{\frac{1}{2}}\right] = \sum_{n=0}^{\infty} P^m_{m+n}(x)\,\frac{t^{m+n}}{(2m+n)!},$$

where

$$I_m(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{m+2n}}{n!\,(m+n)!},$$

Combining this result with (2.2) and using the orthogonal property of associated Legendre functions, we have

$$(3.6) \qquad \int_{-1}^{1} (1+x)^{(m+n)/2} e^{tx} I_m \left[t(x^2-1)^{\frac{1}{2}} \right] P^m_{m+n} \left(\sqrt{\frac{1+x}{2}} \right) dx$$

$$= \frac{(2m+n)!\, t^m}{2^{(m+n-2)/2}\,(2m+1)!\, n!}\ {}_2F_2 \left[\begin{matrix} -n, m+\frac{1}{2} \\ 2m+1,\ m+3/2 \end{matrix} ; -t \right].$$

With Thosar's [15] relation

$$(3.7) \qquad x^{n-m}(1-x^2)^{m/2} = (n-m)! \sum_{r=0}^{[(n-m)/2]} \frac{(2n-4r+1)\, 2^{n-2r}(n-r)!}{(2n-2r+1)!\, r!}\, P^m_{n-2r}(x),$$

where m is an integer $\leqslant n$, and (2.2), we obtain after using the orthogonal property of Legendre functions

$$(3.8) \qquad \int_{-1}^{1} (1+x)^{(m+n)/2} (1-x)^{m/2} x^{n-m} P^m_{m+n} \left(\sqrt{\frac{1+x}{2}} \right) dx.$$

$$= \frac{(2m+n)!\, n!}{(2n+1)!\, m!\, 2^{(m-n-2)/2}}\ {}_3F_2 \left[\begin{matrix} -n-\frac{1}{2},\ (m-n)/2,\ (m-n+1)/2 \\ (m+1)/2,\ m/2+1 \end{matrix} ; -1 \right]$$

Again we have the relation [11, p. 264]

$$(3.9) \qquad (1-2xt+t^2)^{\nu/2}\, P^{\mu}_{\nu} \left[\frac{x-t}{(1-2xt+t^2)^{\frac{1}{2}}} \right] = \sum_{r=0}^{\infty} \binom{\nu+\mu}{r} P^{\mu}_{\nu-r}(x)\,(-t)^r$$

Taking $\mu = m$, $\nu = n$, m and n being positive integers, we have

$$(1-2xt+t^2)^{n/2}\, P^m_n \left[\frac{x-t}{(1-2xt+t^2)^{1/2}} \right] = \sum_{r=0}^{n-m} \binom{n+m}{r} P^m_{n-r}(x)\,(-t)^r.$$

Combining this result with (2.2) and using (3.1), we have

$$(3.10) \qquad \int_{-1}^{1} (1+x)^{(m+n)/2} (1-2xt+t^2)^{n/2}\, P^m_n \left[\frac{x-t}{(1-2xt+t^2)^{1/2}} \right] P^m_{m+n} \left(\sqrt{\frac{1+x}{2}} \right) dx$$

$$= \frac{(2m+n)!}{2^{(m+n-2)/2}\, m!\,(n-m)!\,(2n+1)}\ {}_3F_2 \left[\begin{matrix} -n-m,\ m-n,\ -n-\frac{1}{2} \\ -n+\frac{1}{2},\ m+1 \end{matrix} ; -t \right].$$

Also we have due to Baily [1]

$$(3.11) \qquad (x^2-1)^{m/2} P_p^m(x) P_q^m(x) = \frac{(p+m)!\,(q+m)!}{2^m\,(p-m)!\,(q-m)!} \times$$

$$\sum_{r=0}^{q+m} \frac{A_{r,m}\, A^m_{p-r}\, A^m_{q-r}\,(2p+2q+2m-4r+1)}{A^{-m}_{(p+q+m-r)}\,(2p+2q+2m-2r+1)} P^m_{p+q+m-2r}(x)$$

where

$$A^m_s = \frac{(\frac{1}{2})_s}{(s+m)!}, \quad A_{r,m} = \frac{(\frac{1}{2}-m)_r}{r!};$$

p, q, m being all positive integers such that $p \geqslant q+2m$, $q \geqslant m$. If $n=p+q$, combining the above relation with (2.2) and using (3.1), we have

$$(2.12) \qquad \int_{-1}^{1} (1+x)^{(m+p+q)/2} (x^2-1)^{m/2}\, P^m_{m+p+q}\left(\sqrt{\frac{1+x}{2}}\right) P^m_p(x)\, P^m_p(x)\, dx$$

$$= \frac{(2m+p+q)!\,(\frac{1}{2})_p\,\Gamma(\frac{1}{2}+q)}{\Gamma(3/2+p+q+m)\,(p-m)!\,(q-m)!\,2^{(3m+p+q)/2}}$$

$${}_6F_5\left[\begin{matrix}\frac{1}{2}-m, -m-p, -m-q, -\frac{1}{2}-p-q-m, -\frac{p+q}{2}, -\frac{p+q-1}{2}, 1 \\ \frac{1}{2}-p, \frac{1}{2}-q, -p-q, \frac{1}{2}, 1\end{matrix}\right].$$

4. SOME GENERALIZED INTEGRALS

We have the integrals [5]

$$(4.1) \qquad \int_{-1}^{1} (1-x^2)^{m/2} (1+x)^p\, P^m_n(x)\, dx$$

$$= \frac{(-1)^{m/2}\, 2^{p+m+1}\,\Gamma(m+n+1)\,\Gamma(p+1)\,\Gamma(p+m+1)}{m!\,\Gamma(n-m+1)\,\Gamma(p+m+n+2)\,\Gamma(p+m-n+1)},$$

where $p > -m-1$, m being a positive integer and n unrestricted ; and

(4.2) $$\int_{-1}^{1} (1-x^2)^{-m/2} (1+x)^p P_n^m(x)\, dx$$

$$=\frac{(-1)^{m/2}\, 2^{p-m+1}\, \Gamma(p+1)\, \Gamma(p-m+1)}{\Gamma(p+n-m+2)\, \Gamma(p-m-n+1)},$$

m and n being unrestricted and $p>m-1$.

Substituting $x=2y^2-1$ in the above two integrals, we have

(4.3) $$\int_{0}^{1} y^{m+2p+1} (y^2-1)^{m/2} P_n^m(2y^2-1) dy$$

$$=\frac{\Gamma(m+n+1)\, \Gamma(p+1)\, \Gamma(p+m+1)}{2\, m!\, \Gamma(n-m+1)\, \Gamma(p+m+n+2)\, \Gamma(p+m-n+1)}$$

$=0$, if $p<n-m$, p and n being integers; and

(4.4) $$\int_{0}^{1} y^{2p-m+1} (1-y^2)^{-m/2} P_n^m(2y^2-1)\, dy$$

$$=\frac{(-1)^{m/2}\, \Gamma(p+1)\, \Gamma(p-m+1)}{2\, \Gamma(p+n-m+2)\, \Gamma(p-n-m+1)},$$

where $p>m-1$, m and n being unrestricted.

For $m=0$, (4.3) and (4.4) reduce to Cooke's result [10].

Next, Shabde [14] has shown that

$$(1-x^2)^{-(m_1+m_2+\dots+m_k)/2}\, P^{m_1}_{p_1+m_1}(x)\, P^{m_2}_{p_2+m_2}(x) \dots\dots P^{m_k}_{p_k+m_k}(x)$$

is a polynomial of degree $p_1+p_2+\dots\dots+p_k=s$ (say).

Now consider the integral

$$I=\int_{0}^{1} (1-x^2)^{-(m_1+m_2+\dots+m_k)/2}\, P^{m_1}_{p_1+m_1}(x) \dots\dots P^{m_k}_{p_k+m_k}(x)\, (x^2-1)^{m/2}\, x^{m+s+1}\, P^m_{m+s}(2x^2-1) dx$$

$$=\int_0^1 (a_0x^s+a_1x^{s-1}+\ldots)\,(x^2-1)^{m/2}\,x^{m+s+1}\,P^{m}_{m+s}\,(2x^2-1)\,dx.$$

On using (4.3), we obtain

$$I=\frac{a_0\,\Gamma(2m+s+1)\,\Gamma(s+m+1)}{2\quad m!\,\Gamma(2m+2s+2)}.$$

Therefore

$$(4.5)\qquad \int_0^1 (-)^{\frac{m}{2}}\,(1-x^2)^{\{m-m_1-m_2-\ldots-m_k\}/2}\,x^{s+m+1}\,P^{m_1}_{p_1+m_1}(x)\ldots P^{m_k}_{p_k+m_k}(x)\,P^{m}_{s+m}(2x^2-1)\,dx$$

$$=\prod_{r=1}^{k}\left\{\frac{(-1)^{m_r}\,(2p_r+2m_r)!}{2^{m_r}\,p_r!\,(p_r+m_r)!}\right\}\frac{\Gamma(2m+s+1)\,\Gamma(s+m+1)}{2^{s+1}\,\Gamma(2m+2s+1)}.$$

5. DERIVATIVE OF THE SERIES

In relation (2.2) replacing m by $2\,m$, we have

$$(5.1)\qquad 2^{m+n/2}\,(1+x)^{m+n/2}\,P^{2m}_{2m+n}\left(\sqrt{\frac{1+x}{2}}\right)=\sum_{r=0}^{n}\binom{4m+n}{n-r}P^{2m}_{2m+r}(x).$$

Bhonsle [4] has shown that

$$(5.2)\qquad \left\{\frac{d^m}{dx^m}\left[P^{2m}_{n}(x)\right]\right\}_{x=1}=\frac{(2m+n)!\,m!}{2^m\,(2m)!\,(n-2m)!}.$$

Therefore, differentiating (5.1) successively m times, we obtain

$$(5.3)\qquad \frac{d^m}{dx^m}\left\{2^{m+n/2}\,(1+x)^{m+n/2}\,P^{2m}_{2m+n}\left(\sqrt{\frac{1+x}{2}}\right)\right\}_{x=1}$$

$$=\frac{(4m+n)!\,m!\,2^n}{2^m\,(2m)!\,n!}.$$

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REFERENCES

1. Baily, W. N., Generalized Hypergeometric Series, 1935, p. 101.

2. Bhonsle, B. R., On a series of Rainville involoing Legendre polynomials, *Proc. Amer, Math, Soc.*, Vol, 8, No. 1, 1957, p. 10-14.

3. Bhonsle, B. R, On some results involving Legendre polynomials, Ganit, Vol. 8, No. 1, 1957, p. 9-16.

4. Bhonsle, B. R., On a problem of Grosswald for Legendre polynomials, *Bull. Cal Math. Soc.*, Vol 49, No. 1, 1957, p. 34-36.

5. Bhonsle, B. R. and Varma, C. B. L., On some integrais involving Legendre polynomials, associated Legendre functions and Jacobi polynomials, *Bull. Call. Math. Sec.*, Vol. 48, No. 2, 1956, p. 103-106.

6. Eloch, E. L., On a expansion of Bessel functions and a series of associated Legendre functions *Akad. Nauk. SSSR. Mat. Meh.*, Vol. 18, 1954, p. 745-748.

7. Carlitz, L., Notes on a formula of Rainville, *Bull. Cal. Math. Soc.*, Vol. 51, 1959, p. 132-133.

8. Chatterjea, S. K., Notes on a formula of Carlitz, Rendiconti de *Sem. Math. Univ. Padova*, Vol. 31, 1961, p. 243-248.

9. Chatterjea, S. K., On a series of Carlitz involving ultraspherical polynomials, Rendiconti de *Sem. Math. Univ. Padova*, Vol. 31, 1961, p. 294-300.

10. Cooke, R. C., *Proc. Lond. Math. Soc.* (2), Vol. 23, 1924, p. 19.

11. Erdelyi, A., Higher Transcendental Functions, Vol. III, McGraw-Hill.

12. Raiville, E. D., Notes on Legendre polynomials, *Bull, Ams. Math. Soc.*, Vol. 51, 1945, p. 268-271.

13. Rangarajan, S. K., *Quar Jour. Math.* Oxford Series (To appear).

14. Shabde, N. G., On some results involving Legendre functions, *Proc. Ben. Math. Sec*, Vol. 7, No. 1, 1945, p. 1-2.

15. Thosar, Y. V., *Math. Zeitschr*, Vol. 60, 1954, p. 52.

16. Yadao, G. M., A new formula for $P^{m}_{m+n}(\cos\alpha)$, *Quar. Jour. Math.* Oxford Series (2), Vol. 13, 1963, p. 29-30.

SOME INTEGRALS INVOLVING A GENERALIZATION OF LOMMEL AND MAITLAND FUNCTIONS, III

By

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ABSTRACT

In previous papers, I have studied the properties of the function,

$$J^{\mu}_{\nu,\lambda}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{\nu+2r+2\lambda}}{\Gamma(1+\lambda+r)\,\Gamma(1+\lambda+\nu+\mu r)}, \quad (\mu>0),$$

which reduces to Lommel's function [1] for $\mu=1$ and to Maitland's function [2] for $\lambda=0$.

The object of this not is to establish some integrals involving this function.

1. In previous papers, I have studied the properties of the function

$$J^{\mu}_{\nu,\lambda}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{\nu+2r+2\lambda}}{\Gamma(1+\lambda+r)\,\Gamma(1+\lambda+\nu+\mu r)}, \quad (\mu>0),$$

which reduces to Lommel's function[1] for $\mu=1$ and to Maitland's function (2) for $\lambda=0$.

The object of this note is to establish some integrals involving this function.

To begin with, we have (Wright[3])

$$\text{(i)} \qquad J^{\mu}_{\nu,\lambda}(x) \sim x^{\nu+2\lambda-2k(\nu+2\lambda+\frac{1}{2})} \exp\left\{ \frac{(\mu x^2/4)^k \cos\pi k}{\mu k} \right\},$$

for large x, where $\mu>0$ and $k=\dfrac{1}{1+\mu}$.

$$\text{(ii)} \qquad J^{\mu}_{\nu,\lambda}(x) \sim x^{\nu+2\lambda-2},$$

for large x, where $0<\mu\leq 1$ and $R(\lambda)$, $R(\nu+\lambda-\mu+1) \neq 0, -1, -2, \ldots$.

2. Using the result ((4), p. 326):

$$\int_0^1 x^{s-1}(1-x^2)^{-\mu/2} P^{\mu}_{\nu}(x)\,dx = \pi^{1/2}\, 2^{\mu-s} \frac{\Gamma(s)\,\Gamma(s/2+1/2-\nu/2-\mu/2)}{\Gamma(1+s/2+\nu/2-\mu/2)}.$$

where $R(\mu) < 1$ and $R(s) > o$, to integrate term by term,

we have

$$\int_o^1 x^\rho (1-x^2)^{-\mu/2} \; P_\nu^\mu(x) \; J_{\beta,\gamma}^\alpha (px^{1/2}) \, dx$$

$$= \frac{\pi^{1/2} \; p^{\beta+2\gamma}}{p^{3.2} \; \beta+3\gamma+\rho-\mu+1} \times$$

$$\times \sum_{r=o}^{\infty} \frac{(-(p^2/8)^r \; \Gamma(\beta/2+\rho+\gamma+r+1) \; \Gamma(\beta/4+\rho/2+\gamma/2+r/2+1-\nu/2-\mu/2)}{\Gamma(1+\gamma+r) \; \Gamma(1+\gamma+\beta+\alpha r) \; \Gamma(\beta/4+\rho/2+\gamma/2+r/2+3/2+\nu/2-\mu/2)}, \quad (2.1)$$

where $p > o$, $\alpha > o$, $R(\mu) < 1$ and $R(\beta+2\rho+2\gamma) > -2$.

Particular Case : If $\rho = -\beta/2$, $\alpha = \frac{1}{2}$ and $\gamma = -2\beta-\nu-\mu$, then

$$\int_o^1 x^{-\beta/2} (1-x^2)^{-\mu/2} \; P_\nu^\mu(x) \; J_{\beta,-2\beta-\nu-\mu}^{\frac{1}{2}} (px^{1/2}) \, dx$$

$$= \pi^{1/2} \frac{2^{4\mu+5\beta+3\nu-1}}{p^{3\beta+2\nu+2\mu}} E_{1/2,\, 3/2-\beta-\mu} (-p^2/8) \quad (2.1.1)$$

where $p > o$, $R(\mu) < 1$, $R(\nu+\mu+2\beta) < 1$ and $E_{\alpha,\beta}$ is the generalized Mittag-Leffler's function (5).

3. In a similar way, using the result ((4). p. 326):

$$\int_o^1 x^{s-1} (1-x^2)^{m/2} \; P_\nu^m(x) \, dx$$

$$\pi^{1/2} (-1)^m \, 2^{-m-s} \, \Gamma(s) \; \frac{\Gamma(1+m+\nu) \; \Gamma(1/2+s/2+m/2-\nu/2)}{\Gamma(1-m+\nu) \; (1+s/2+m/2+\nu/2)},$$

where $R(s) > o$, we may prove that

$$\int_o^1 x^{-\beta/2} (1-x^2)^{m/2} \; P_\nu^m(x) \; J_{\beta,-2\beta+m-\nu}^{1/2} (px^{1/2}) \, dx$$

$$= \pi^{1/2} \frac{2^{5\beta-4m+3\nu-1}}{p^{3\beta-2m+2\nu}} (-1)^m \frac{\Gamma(1+m+\nu)}{\Gamma(1-m+\nu)} E_{1/2,\, 3/2+m-\beta} (-p/8), \quad (3.1)$$

where $p > o$ and $R(-2\beta+m \cdot \nu) > -1$.

4. Using the result ((4), p. 351)

$$\int_0^2 x^{s-1} (4-x^2)^{-1/2} T_n(1/2x)\, dx$$

$$= \pi/2 \frac{\Gamma(s)}{\Gamma^*(1/2+s/2\pm n/2)}, \ (R(s) > o),$$

to integrate term by term, we have

$$\int_0^2 x^{\rho} (4-x^2)^{-1/2} T_n(1/2x)\ J^{\mu}_{\nu,\lambda}\ (ax)\, dx$$

$$= \pi/2 \sum_0^{\infty} \frac{(-1)^r (a/2)^{\nu+2r+2\lambda}\ \Gamma(1+\rho+\nu+2r+2\lambda)}{\Gamma(1+\lambda+r)\ \Gamma(1+\lambda+\nu+\mu r)\ \Gamma_*(1\pm n/2+\rho/2+\nu/2+r+\lambda)}, \quad (4.1)$$

where $a > o$, $R(\nu+\rho+2\lambda) > -1$ and $\mu > o$.

In particular, for $\rho=\lambda=o$ and $\mu=2$, we have

$$\int_0^2 x^{\nu} (4-x^2)^{-1/2}\ T_n(1/2x)\ J^2_{\nu}(-x^2a^2/4)\, dx$$

$$= \pi a^{(\nu-2)/3}\ 2^{(2\nu-1)/3}\ J_{1/2+n/2+\nu/2,1/2-n/2+\nu/2}\left(3\,(a/2)^{2/3}\right), \quad (4.1.1)$$

where $a > o$ and $R(\nu) > o$,

5. The termwise integration leads to the integral

$$\int_0^{\infty} t^{p-1} e^{-at^2}\ J^{\mu}_{\nu,\lambda}\ (bt)\,dt$$

$$= \frac{(b/2)^{\nu+2\lambda}\ \Gamma\left(\frac{\nu+p}{2}+\lambda\right)}{2a^{\frac{\nu+p}{2}+\lambda}\ \Gamma(1+\lambda)\ \Gamma(1+\lambda+\nu)} \times$$

$$\times\ {}_2F_{\mu+1}\left(\begin{matrix} 1, \frac{\nu+p}{2}+\lambda & ; \frac{-b^2}{4a}\ \mu^{-\mu} \\ 1+\lambda,\ \frac{1+\lambda+\nu}{\mu},\ \frac{2+\lambda+\nu}{\mu}, \ldots, \frac{\mu+\lambda+\nu}{\mu} & \end{matrix}\right), \quad (5.1)$$

where $a > o$, $b > o$, $R(\nu+p+2\lambda) > o$ and μ is a positive integer.

In particular, for $\mu=1$, we have

$$\int_0^\infty t^{p-1} e^{-\alpha t^2} s_{\sigma,\nu}(bt)\, dt$$

$$= \frac{b^{\sigma+1}\,\Gamma\left(\frac{\sigma+p+1}{2}\right)}{2a^{\frac{1}{2}(\sigma+p+1)}\,(\sigma-\nu+1)(\sigma+\nu+1)}\; {}_2F_2\left(\begin{matrix}1,\ \frac{\sigma+p+1}{2} \\ \frac{\sigma-\nu+3}{2},\ \frac{\sigma+\nu+3}{2}\end{matrix}\ ;\ \frac{-b^2}{4a}\right), \tag{4.1.1}$$

where $a>o, b>o$, and $R(\sigma+p)>-1$.

6. Using Goldstein's integral

$$\int_0^\infty x^{l-1} e^{-\frac{1}{2}x} W_{k,m}(x)\, dx = \frac{\Gamma_*(l\pm m+\frac{1}{2})}{\Gamma(l-k+1)},$$

where $R(l\pm m+\frac{1}{2})>o$.

to integrate term by term, we have

$$\int_0^\infty x^{p-1} e^{-\frac{1}{2}x^2} W_{\rho,\sigma}(x^2)\, J^{\mu}_{\nu,\lambda}(ax)\, dx$$

$$=\tfrac{1}{2}(a/2)^{\nu+2\lambda}\, \frac{\Gamma_*(\nu/2+p/2+1/2+\lambda\pm\sigma)}{\Gamma(1+\lambda)\,\Gamma(1+\lambda+\nu)\,\Gamma(\nu/2+p/2+\lambda-\rho+1)}\times$$

$$\times\ {}_3F_{\mu+2}\left(\begin{matrix}1,\ \nu/2+p/2+1/2+\lambda+\sigma,\ \nu/2+p/2+1/2+\lambda-\sigma \\ 1+\lambda,\ \nu/2+p/2+\lambda-\rho+1,\ (1+\nu+\lambda)/\mu,\ (2+\nu+\lambda)/\mu,\ldots,(\mu+\nu+\lambda)/\mu\end{matrix}\ ;\ -a^2/4\,\mu^{-\mu}\right), \tag{6.1}$$

where $a>o$, $R(\nu+p+1+2\lambda\pm2\sigma)>o$ and μ is a + ve integer,

Particular Cases

(i) $\rho=o$:

Using the formula

$$W_{o,\mu}(x)=\sqrt{\left(\frac{x}{\pi}\right)}K_\mu\left(\frac{x}{2}\right),$$

we get

$$\int_0^\infty x^{p} - e^{\frac{1}{2}x^2}\, K\sigma(x^2/2) J^{\mu}_{\nu,\lambda}(ax)\, dx$$

$$= \frac{\sqrt{\pi}}{2}(a/2)^{\nu+2\lambda} \frac{\Gamma_*(\nu/2+p/2+1/2+\lambda\pm\sigma)}{\Gamma(1+\lambda)\ \Gamma(1+\lambda+\nu)\ \Gamma(\nu/2+p/2+\lambda+1)} \times$$

$$\times {}_3F_{\mu+2}\left(\begin{matrix}1,\ \nu/2+p/2+1/2+\lambda+\sigma,\ \nu/2+p/2+1/2+\lambda-\sigma & ;-a^2/4\mu^{-\mu} \\ 1+\lambda,\ 1+\lambda+\nu/2+p/2,\ (1+\lambda+\nu)/\mu,\ (2+\lambda+\nu)/\mu,\ldots,\ (\mu+\lambda+\nu)/\mu & \end{matrix}\right)$$ (6.1.1)

where $a>0$, $R(\nu+p+1+2\lambda\pm2\sigma)>0$ and μ is a positive integer,

(ii) $\sigma=\frac{1}{4}$:

Using the formula

$$W_{\mu,\pm}(x)=2^{-\mu}\ (2x)^{\frac{1}{4}}\ D_{2\mu-\frac{1}{2}}\ [(2x^{1/2})],$$

we get

$$\int_0^\infty x^{p-\frac{1}{2}}e^{-\frac{1}{2}x^2}\ D_{2\rho-\frac{1}{2}}\ (x\sqrt{2})\ J^{\mu}_{\nu,\lambda}\ (ax)\ dx$$

$$= (a/2)^{\nu+2\lambda}\ 2^{\rho-5/4} \frac{\Gamma(\nu/2+p/2+\lambda+3/4)\ \Gamma(\nu/2+p/2+\lambda+1/4)}{\Gamma(1+\lambda)\ \Gamma(1+\lambda+\nu)\ \Gamma(\nu/2+p/2+\lambda-\rho+1)} \times$$

$$\times {}_3F_{\mu+2}\left(\begin{matrix}1,\ \nu/2+p/2+\lambda+3/4,\ \nu/2+p/2+\lambda+1/4) & ;-a^2/4\ \mu^{-\mu} \\ 1+\lambda,\ 1+\lambda+\nu/2+p/2-\rho,\ (1+\lambda+\nu)/\mu,\ (2+\lambda+\nu)/\mu,\ldots,\ (\mu+\lambda+\nu)/\mu) & \end{matrix}\right),$$ (6.1.2)

where $R(\nu+p+2\lambda)>-\frac{1}{2}$, $a>0$ and μ is a + ve integer.

(a) $\mu=1$:

$$\int_0^\infty x^{\rho}\ e^{-x^2/4}\ D_{-\alpha}\ (x)\ s_{\nu+2\lambda-1,\nu}\ (ax)\ dx$$

$$= a^{\nu+2\lambda} \frac{2^{\rho/2+\nu/2-\alpha/2+\lambda-5/2}\ \Gamma(\rho/2+\nu/2+1/2+\lambda)\ \Gamma(\rho/2+\nu/2+1+\lambda)}{\lambda\ (\nu+\lambda)\ \Gamma(\rho/2+\nu/2+\alpha/2+\lambda+1)} \times$$

$$\times {}_3F_3\left(\begin{matrix}1,\ \rho/2+\nu/2+1/2+\lambda,\ \rho/2+\nu/2+\lambda+1 \\ 1+\lambda,\ 1+\lambda+\nu,\ 1+\lambda+\ \rho/2+\nu/2+\alpha/2\end{matrix};-a^2/2\right),$$ (6.1.2a)

where $a>0$ and $R(\nu+2\lambda+\rho)>-1$).

(b) $\mu=2$:

$$\int_0^\infty x^{\nu+\alpha}e^{-\frac{1}{4}x^2}\ D_{-\alpha}\ (x)\ J^{\,2}_{\nu,-\nu-\alpha}\ (ax)\ dx$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{2^{\alpha/4-\nu/2}}{a^{\alpha}}\ J_{-\nu-\alpha}\left(\frac{a}{\sqrt{2}}\right),$$

where $R(\alpha)<1$ and $a>0$. (6.1.2b)

(iii) $\rho=\frac{1}{2}n+\frac{1}{4}$, $\sigma=\frac{1}{4}$:

Using the formula

$$W_{\frac{1}{2}n+\frac{1}{4},\pm\frac{1}{4}}(x) = 2^{-\frac{1}{2}n}\, x^{\frac{1}{4}}\, e^{-\frac{1}{2}x}\, \mathrm{He}n\left[(2x)^{\frac{1}{2}}\right],$$

where Hen (x) is the Hermite polynomial, we get

$$\int_0^\infty x^{p-\frac{1}{2}}\, e^{-x^2}\, \mathrm{He}n\,(x\sqrt{2})\, J^{\mu}_{\nu,\lambda}(ax)\, dx$$

$$= 2^{\frac{1}{2}n-1}\,(a/2)^{\nu+2\lambda}\,\frac{\Gamma(\nu/2+p/2+\lambda+3/4)\,\Gamma(\nu/2+p/2+\lambda+1/4)}{\Gamma(1+\lambda)\,\Gamma(1+\lambda+\nu)\,\Gamma(\nu/2+p/2+\lambda-n/2+3/4)}\times$$

$$\times\ {}_3F_{\mu+2}\left(\begin{matrix}1,\ \nu/2+p/2+\lambda+3/4,\ \nu/2+p/2+\lambda+1/4 & ;\ -a^2/4\ \mu^{-\mu}\\ 1+\lambda,\ \nu/2+p/2-n/2+\lambda+3/4,\ (1+\lambda+\nu)/\mu,\ (2+\lambda+\nu)/\mu,\ldots,\ (\mu+\lambda+\nu)/\mu & \end{matrix}\right), \qquad (6.1.3)$$

where $a>o$, $R(\nu+p+2\lambda)>-\frac{1}{2}$ and μ is a positive integer.

(iv) $\rho=\sigma+n+\frac{1}{2}$:

Using the formula

$$W_{\sigma+n+\frac{1}{2},\pm\sigma}(x)=(-1)^n\, n!\, x^{\sigma+\frac{1}{2}}\, e^{-\frac{1}{2}x}\, L_n^{2\sigma}(x),$$

where $L_n^{\alpha}(x)$ is the Laguerre polynomial, we get

$$\int_0^\infty x^{p+2\sigma}\, e^{-x^2}\, L_n^{2\sigma}(x^2)\, J^{\mu}_{\nu,\lambda}(ax)\, dx$$

$$=\frac{(-1)^n}{n!}\,(a/2)^{\nu+2\lambda}\,\frac{\Gamma_*(\nu/2+p/2+1/2+\lambda\pm\sigma)}{\Gamma(1+\lambda)\,\Gamma(1+\lambda+\nu)\,\Gamma(\nu/2+p/2+\lambda-\sigma-n+1/2)}\times$$

$$\times\ {}_3F_{\mu+2}\left(\begin{matrix}1,\ \nu/2+p/2+1/2+\lambda+\sigma,\ \nu/2+p/2+1/2+\lambda-\sigma & ;\ -a^2/4\ \mu^{-\mu}\\ 1+\lambda,\ \nu/2+p/2+\lambda-\sigma-n+1/2,\ (1+\lambda+\nu)/\mu,\ (2+\lambda+\nu)/\mu,\ldots,\ (\mu+\lambda+\nu)/\mu & \end{matrix}\right) \qquad (1.1.4)$$

where $a>o$, $R(\nu+p+2\lambda\pm2\sigma)>-1$ and μ is a +ive integer.

(v) $\mu=1$:

Using the relation

$$J^{1}_{\nu,\lambda}(x)=\frac{2^{2-\nu-2\lambda}}{\Gamma(\lambda)\,\Gamma(\nu+\lambda)}\, s_{\nu+2\lambda-1,\nu}(x),$$

we get

$$\int_0^\infty x^{p-1}\, e^{-\frac{1}{2}x^2}\, s_{\nu+2\lambda-1,\nu}(ax)\, W_{\rho,\sigma}(x^2)\, dx$$

$$=\frac{a^{\nu+2\lambda}}{8}\,\frac{\Gamma_*(\nu/2+p/2+1/2+\lambda\pm\sigma)}{\lambda(\nu+\lambda)\,\Gamma(\nu/2+p/2+\lambda-\rho+1)}\times$$

$$\times\ {}_3F_3\left(\begin{matrix}1,\ \nu/2+p/2+1/2+\lambda\pm\sigma, & \\ 1+\lambda,\ 1+\lambda+\nu,\ \nu/2+p/2+\lambda-\rho+1 & ;\ -a^2/4\end{matrix}\right), \qquad (6.1.5)$$

where $a>0$ and $R(\nu+p+1+2\lambda\pm\sigma)>0$

(vi) $\lambda=0$:

Using the relation

$$J^{\mu}_{\nu,0}\ (x)=(x/2)^{\nu}\ J^{\mu}_{\nu}\ (x^2/4),$$

where $J^{\mu}_{\nu}\ (x)$ is the Bessel Maitland function, we get

$$\int_0^{\infty} x^{\nu+p-1}\ e^{-\frac{1}{2}x^2}\ W_{\rho,\sigma}\ (x^2)\ J^{\mu}_{\nu}\ (x^2a^2/4)\ dx$$

$$=\tfrac{1}{2}\frac{\Gamma_*(\nu/2+p/2+1/2\pm\sigma)}{\Gamma(1+\nu)\ \Gamma(\nu/2+p/2-\rho+1)}\times$$

$$\times\,{}_2F_{\mu+1}\left(\begin{matrix}\nu/2+p/2+1/2\pm\sigma\\ \nu/2+p/2-\rho+1,\ (1+\nu)/\mu,\ (2+\nu)/\mu,\ldots,\ \mu+\nu)/\mu\end{matrix}\ ;\ -a^2/4\ \mu^{-\mu}\right) \tag{6.1.6}$$

where $a>0$, $R(\nu+p+1\pm2\sigma)>0$
and μ is a $+$ve integer.

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REFERENCES

(1) Watson, G. N.: Theory of Bessel functions (1958), p. 345.

(2) Wright, E. M.: *J. London Math. Soc.* (1933), 8, 71-79.

(3) ————————: *J. London Math. Soc.* (1935), 10, 287-293.

(4) Erdelyi, A.: Tables of Integral Transforms Vol. I (1954).

(5) ————————: Higher Transcendental Function; Vol. III (1955). p. 210.

A CLASS OF INTEGRAL EQUATIONS INVOLVING JACOBI POLYNOMIALS AS KERNEL*

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1. INTRODUCTION

Recently, Ta Li [12] has obtained an inversion integral for integral equation involving Chebyshey polynomial as kernel. Similar problems with Legendre, associated Legendre and ultraspherical polynomials have been solved by Ruschman [1, 2, 3], Erdelyi [4], Higgins [7] and Widder [13]. The author [8] by the application of Mellin transformation has obtained inversion integrals for integral equations involving Jacobi polynomials as kernel. The integral equations considered in [8] are not singular. In this note by the application of fractional integration, inversion integrals for integral equations, which are singular, have been obtained. The solutions of these integral equations are given in the form of singular integrals involving Jacobi polynomials. At another place [9] inversion integrals for singular integral equations involving ultraspherical polynomials as kernel have been obtained. The results of Ta Li [12] and of the author [9] are particular cases of the results given here. For the purpose of this note, Jacobi polynomials are defined by

$$(1)\quad M(n,\alpha,\beta,x)=\frac{P_n^{(\alpha,\beta)}(2x^2-1)}{P_n^{(\alpha,\beta)}(1)}={}_2F_1[-n,\ n+\alpha+\beta+1;\ \alpha+1;\ 1-x^2].$$

2. NOTATION

Throughout this note the following notations are used

$$F(u,\sigma)=(u^2-\sigma^2)^{\alpha}\ M(n,\alpha,-\beta,u/\sigma),$$

$$G(u,v)=(v^2-u^2)^{-\alpha-1}\ M(n-1,-\alpha-1,\beta,u/v),$$

$$F_1(u,\sigma)=(u^2-\sigma^2)^{\alpha}\ M(n,\alpha,\beta,u/\sigma),$$

$$G_1(u,v)=(v^2-u^2)^{-\alpha-1}M(n,-\alpha-1-\beta,u/v),$$

$$H(v)=\left\{-v^{-1}d/dv\right\}^{2n}\left\{v^{2n}f_{2n}(v)\right\},$$

$$H_1(v)=\left\{-v^{-1}d/dv\right\}^{2n+1}\left\{v^{2n+1}f_{2n+1}(v)\right\},$$

$$Z(v)=F_{2\beta-2\alpha-1,\,2n-1}\{H(v)\},$$

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$$Z_1(v) = F_{1-2\alpha-2\beta,\ 2n}\{H_1(v)\},$$

$$A = 2\sin(\pi\alpha)\}/\pi.$$

3. THE OPERATOR $F_{\lambda,n}$

The fractional integral of order μ is defined by the equation [5, p. 181].

$$g(y,\mu) = \frac{1}{\Gamma(\mu)}\int_y^a (x-y)^{\mu-1} f(x)\,dx.$$

We make use of this definition and introduce the operators $F_{\lambda,n}$ that occur in this not. The operator $F_{\lambda,n}$ is defined by the formula

$$(2)\ F_{\lambda,n}\left\{f(x)\right\} = \frac{x^{-\lambda-n}}{2^{n-1}\,\Gamma(n)}\int_x^1 (y^2-x^2)^{n-1}\, y^{\lambda} f(y)\,dy.$$

where n is a positive integer and λ is unrestricted, $x>0$.

Since n is a positive integer, by explicit computation, we have

$$(3)\ \left\{\frac{d}{xdx}\right\}\left\{x^{\lambda+n} F_{\lambda,n}\{f(x)\}\right\} = -\,x^{\lambda+n-1} F_{\lambda,n-1}\{f(x)\},$$

$$(4)\ \frac{d}{dx}\left[\left\{\frac{d}{xdx}\right\}^{n-1}\left\{x^{\lambda+n} F_{\lambda,n}\{f(x)\}\right\}\right] = (-)^n x^{\lambda} f(x),$$

$$(5)\ F_{\lambda,n}\left\{f(x)\right\} = 0 \text{ for } x = 1.$$

4. INTEGRAL EQUATIONS AND THEIR SOLUTIONS

Consider the integral equations

$$(6)\ \int_\sigma^1 F(u,\sigma)\, y_{2n}(u)\,du = f_{2n}(\sigma) \quad \sigma\varepsilon I,\ n=1,2,3\ldots$$

and

$$(7)\ \int_\sigma^1 u\,F_1(u,\sigma)\, y_{2n+1}(u)\,du = \sigma f_{2n+1}(\sigma),\ \sigma\varepsilon I,\ n=1,2,3\ldots$$

where $I \equiv \{\sigma : c \leqslant \sigma \leqslant 1\}$, $c>0$ is a constant, $f_{2n}(\sigma)$ and $f_{2n+1}(\sigma)$ are defined on I. It is assumed that

(a) $-1<\alpha<0,\ -1<\beta<1,$

(b) $f_{2n}^{(k)}(1) = 0$ and $f_{2n+1}^{(m)}(1) = 0$ for $0 \leqslant k \leqslant 2n-1$, $0 \leqslant m \leqslant 2n$,

(c) $f_{2n}^{(k)}(\sigma)$ and $f_{2n+1}^{(m)}(\sigma)$ are piecewise continuous on I.

We prove the following theorem.

Theorem 1 :—Given $f_{2n}(\sigma)$ and $f_{2n+1}(\sigma)$ on I satisfying the conditions (b) and (c), then the solutions of (6) and (7) are given by

$$(8)\ y_{2n}(u) = A \int_u^1 (uv)\, G(u, v)\, Z(v)\, dv,$$

and

$$(9)\ y_{2n+1}(u) = A \int_u^1 v^2\, G_1(u, v)\, Z_1(v)\, dv,$$

respectively, provided the condition (a) is satisfied.

5. TWO KEY INTEGRALS

The above theorem will be proved with the help of the following integrals

$$(10) \int_\sigma^v F(u, \sigma)\, G(u, v)\, u\, du = -A^{-1} (\sigma/v)^{2n-2}\, {}_2F_1 \left[-2n+1, \alpha-\beta+2; 1; 1-\frac{v^2}{\sigma^2}\right]$$

$$= -\frac{\sigma^{-2n}}{A.\, 2^{2n-1}.\, \Gamma(2n)}\, v^{2(n-\alpha+\beta-1)} \left\{ \frac{d}{vdv} \right\}^{2n-1} \left\{ (v^2-\sigma^2)^{2n-1}\, v^{2(\alpha-\beta+1)} \right\}$$

and

$$(11) \int_\sigma^v F_1(u, \sigma)\, G_1(u, v)\, u\, du = -A^{-1} (\sigma/v)^{2n}\, {}_2F_1 \left[-2n, \alpha+\beta+1; 1; 1-\frac{v^2}{\sigma^2}\right]$$

$$= -\frac{\sigma^{-2n}}{A.2^{2n}\, \Gamma(2n+1)},\, v^{2(n-\alpha-\beta)} \left\{ \frac{d}{vdv} \right\}^{2n} \left\{ (v^2-\sigma^2)^{2n}\, v^{2\alpha+2\beta} \right\}$$

for $n \geqslant 1$, $-1 < \alpha < 0$, $-1 < \beta < 1$.

This type of integrals have been evaluated by the author at another place [10, 11]. The integral (10) obtained by using the equations (1), (6), (8) and (10) of [10]. Similarly the integral (11) is obtained by using (2),(7),(9) and (10) of [10]. Rodrigue's type formulae in (10) and (11) are obtained by using [6, p. 102 (17)]. An outline of the the proof of the summation results that have been used in [10] for evaluating the integrals has been given in [11].

6. PROOF OF THE DUAL RELATIONS

On grounds of (a), (b) and (c) it can be shown that the integral (8) exists and that the double integral

$$J = A\int_{\sigma}^{1} F(u, \sigma)\left(\int_{u}^{1} (uv)\, G(u, v)\, Z(v)\, dv\right) du$$

obtained by directly substituting (8) in (6) is convergent. This double integral can be writien as

$$J = \lim_{\varepsilon \to 0} A\int_{\sigma+\varepsilon}^{1-\epsilon} \int_{u+\varepsilon}^{1} (uv)\, F(u, \sigma)\, G(u, v)\, Z(v)\, dv\, du$$

Since the integrals are uniformly bounded and the integrand remains finite in the region

$$R : \begin{cases} u + \epsilon \leqslant v \leqslant 1 \\ \sigma + \epsilon \leqslant u \leqslant 1, \end{cases}$$

it is justifiable to interchange the order of integration in R. Thus we obtain

$$J = \lim_{\epsilon \to 0} A\int_{\sigma+2\epsilon}^{1} v.\, Z(v)\, dv \int_{\sigma+\epsilon}^{v-\epsilon} F(u, \sigma)\, G(u,v)\, u$$

Since M $(n, \alpha, -\beta, u/\sigma)$ M $(n-1, -\alpha-1, \beta, u/v)$ is continuous and finite in $v \leqslant u \leqslant \sigma$ and the integrals

$$\int_{\sigma}^{v} (u^2-\sigma^2)^{\alpha}\, (v^2-u^2)^{-\alpha-1} du \text{ and } \int_{\sigma}^{v} F(u, \sigma)\, G(u, v)\, u\, du$$

exist, hence one can write

$$J = A\int_{\sigma}^{1} v,\, Z(v)\, dv \int_{\sigma}^{v} F(u, \sigma)\, G(u, v)\, u\, du.$$

After substituting the value of the inner integral from (10), one finds that for $n \geqslant 1$

$$J = -\frac{\sigma^{-2n}}{2^{2n-1}\Gamma(2n)} \int_{\sigma}^{1} \frac{d}{dv}\left[\left\{\frac{d}{vdv}\right\}^{2n-2} \left\{(v^2-\sigma^2)^{2n-1}\, v^{2\alpha-2\beta+2}\right\}\right] \left\{ v^{2(n-\alpha+\beta-1)}\, Z(v)\right\} dv.$$

Successive integrations by parts and the application of the operational relations (3), (4) and (5) then yields

$$J = -\frac{\sigma^{-2n}}{2^{2n-1}\,\Gamma(2n)} \int_{\sigma}^{1} (v^2-\sigma^2)^{2n-1}\, d\left[\left\{\frac{-d}{xdv}\right\}^{2n-1} \left\{v^{2n} f_{2n}(v)\right\}\right],$$

Further successive integrations by parts and the application of the conditions $f^{(k)}(1) = 0$, $0 \leqslant k \leqslant 2n-1$, finally yields

$$J = f_{2n}(\sigma).$$

The dual relation (7) and (9) can be verified in the same manner with the help of (11).

7. In this section, it will be shown that Rodrigue's formula for Jacobi polynomial

$$(12) \quad F(u) = (u-1)^k \; u^{\beta} \; P_n^{(k,\beta)}(2u-1) = \{1/(n)!\} \; (d/du)^n \left\{ (u-1)^{n+k} \; u^{n+\beta} \right\}$$

where n and k are integers with $0 \leqslant k \leqslant n$ and $\beta > -1$, leads to a solution of the integral equation

$$(13) \quad \int_1^x F(x/t)\, g(t)\, dt = f(x), \; 1 < x < x_0$$

directly. We actually prove the theorem :

Theorem 2 :—Let f be absolutely continuous on $[1, x_0]$ for some $x_0 > 1$, and $f^{(m)}(1) = 0$ for $0 \leqslant m \leqslant k$; then the solution of (13) is given by

$$(14) \quad g(x) = \frac{n}{(n+k)!} \cdot x^{n+k+\beta} \; (d/dx)^{n+k+1} \left\{ \int_1^x (x-t)^{n-1} \cdot x^{-n-\beta} f(t)\, dt \right\}$$

where n and k are integers with $0 \leqslant k \leqslant n$ and $\beta > -1$.

Proof :—When $n = 0$, clearly $g(x) = \{1/(k)!\} \; x^{n+k} \; (d/dx)^{k+1} \left\{ f(x)/x^{\beta} \right\}$.

We shall assume $n \geqslant 1$. On account of (12), the integral equation (13) can be re-written as

$$(d/dx)^n \left\{ \int_1^x (x-t)^{n+k} \cdot x^{n+\beta} \cdot t^{-n-k-\beta} \; g(t)\, dt \right\} = (n)!\, f(x)$$

by n repeated integrations, it follows that

$$\int_1^x (x-t)^{n+k} \cdot x^{n+\beta} \; t^{-n-k-\beta} \; g(t)\, dt = n. \int_1^x (x-t)^{n-1} f(t)\, dt$$

since the left hand side together with its first $n-1$ derivatives vanishes when $x = 1$.

The above integral can be re-written as

$$\int_1^x (x-t)^{n+k} \cdot t^{-n-k-\beta} \; g(t)\, dt = n. \int_1^x (x-t)^{n-1} \cdot x^{-n-\beta} f(t)\, dt.$$

Differentiating $n+k+1$ times with respect to x we finally, obtain

$$(n+k)!.\, x^{-n-k-\beta}.\, g(x) = n.\, (d/dx)^{n+k+1} \left\{ \int_1^x (x-t)^{n-1} .\, x^{-n-\beta} f(t)\, dt \right\}.$$

It is easy to conform that g given by (14) does indeed satisfy (13) under the conditions on f.

REFERENCES

1. R. G. Buschman, An inversion integral for a Legendre transformation, *Amer, Math Monthly.*, 69 (1962) 288.
2. ———, An inversion integral for a general Legendre transformation, SIAM 5 (1963) 232.
3. ———, An inversion integral, *Proc Amer, Math, Soc.* 13 (1962) 675.
4. A Erdeil, An integral equation involving Legendre's polynomial, *Amer. Math. Monthly*, 70 (19..) 6:1.
5. ———, Table of integral transforms, Vol 2, McGraw— Hill, New York, (1954).
6. ———, Higher transcendental functions, Vol. 1, McGraw—Hill, New York, (1953).
7. T. P. Higgins, An inversion integral for a Gegenbauer transformation, *J. Soc. Indust. Appl. Math*, 11 (1963) 886.
8. K. N. Srivastava, Inversion integerals involving Jacobi polynomials, *Proc. Amer, Math. Soc.* 15 (1964) 635-6.8.
9. ——— A class of integral equations involving ultraspherical polynomilas as kernel, *Proc. Amer. Math Soc.*, 14 (1963) 932. Also *see* the note on this paper in Math. Jap., 9 (1964) 83-84.
10. ———, On some integrals involving Jacobi's polynomials, *Math. Zeitschr*, 82 (1963) 299.
11. ———, On some integrals involving Jocobi's polynomials, *Math. Zeitsohr*, 84 (1964) 257-259.
12, Ta Li, A new class of integral transforms, *Proc. Amer. Math. Soc.* 11 (1960) 288.
15. D. V. Widder, Two convolution transforms which are inverted by convolutions, *Proc. Amer. Math. Soc.* 14 (1963) 812.

CERTAIN INTEGRAL REPRESENTATIONS FOR MODIFIED BESSEL FUNCTION OF SECOND KIND

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ABSTRACT

In the present note, we have evaluated some infinite integrals involving the product of Bessel function of first kind and generalized hypergeometric functions, such as S_4 and MacRobert's E-Functions whose variables are of the type (p^2+t^2), where $|\arg p^2| < \pi$, with the help of Operational Calculus. The integrals have been evaluated by the application of a theorem on Laplace Transform given earlier by me [1964]. In the obtained integrals, when p tends to zero, we get known result of Erdelyi. A. A few integral representations for the Modified Bessel function of Second kind $K_\nu(x)$ have also been obtained as their particular cases, which are believed to be new.

1. The object of this paper is to evalute some infinite integrals involving the product of Bessel function and some other generalized hypergeometric functions by the application of a theorem. Some integral representations for $K_\nu(x)$ have also been obtained.

Throughout this paper, the conventional notation

$$\phi(p) \doteq h(t),$$

will be used to denote the Laplace's integral

$$\phi(p) = p\int_0^\infty e^{-pt}\, h(t)\, dt, \tag{1}$$

provided that $R(p)>0$ and the integral is convergent.

Here we shall make use of a theorem recently given by me [1964] in the following form, that if

$$\phi(p) \doteq h(t),$$

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and

$$\Psi(p) \doteq t^{-\nu-1} e^{-b/t} h(t) ,$$

then $$\Psi(p) = \frac{2p}{b^{\nu/2}} \int_0^\infty t^{\nu+1} (p+t^2)^{-1} J_\nu (2\sqrt{bt})\, \phi(p+t^2)\, dt , \qquad (2)$$

provided that the integral is convergent and the Laplace transforms of $|h(t)|$ and $|t^{-\nu-1} e^{-b/t} h(t)|$ exist, $R(\nu) > -1$, $b>0$ and $|\arg p| < \pi$.

Now we shall evaluate some infinite integrals by the application of the above theorem.

2. *Example I.* Taking [Saxena, 1960 p. 402 (11)]

$$h(t) = t^{-\rho} K_\mu (b/t)$$

$$\doteq \frac{1}{b\sqrt{\pi}} (p/2)^{\rho-1} S_4 (1-\rho/2, \tfrac{1}{2}-\rho/2, \mu/2, -\mu/2 ; bp/4)$$

$$= \phi(p) ,$$

where $R(p)>0$ and $R(b)>0$.

Therefore also

$$t^{-\nu-1} e^{-b/t} h(t) = t^{-\nu-\rho-1} e^{-b/t} K_\mu (b/t)$$

$$\doteq \sqrt{\pi}\, p^{1+\nu+\rho}\, G^{30}_{13} \left(2bp \middle| {\tfrac{1}{2} \atop -\rho-\nu, \mu, -\mu} \right)$$

$$= \Psi(p) ,$$

where $R(p)>0$ and $R(b)>0$.

Using the above values of $\phi(p)$ and $\Psi(p)$ in (2) and replacing b by $\frac{1}{4}b^2$ and p by p^2, we obtain

$$\int_0^\infty t^{\nu+1} (p^2+t^2)^{\rho-2} J_\nu (bt)\, S_4 \left(\tfrac{1}{2}-\rho/2, 1-\rho/2, \mu/2, -\mu/2 ; \frac{b^2(p^2+t^2)}{16} \right) dt$$

$$= \frac{\pi b^{\nu+2}\, p^{2\rho+2\nu}}{2^{\nu-\rho+4}}\, G^{30}_{13} \left(\frac{b^2p^2}{2} \middle| {\tfrac{1}{2} \atop -\rho-\nu, \mu, -\mu} \right) , \qquad (3)$$

valid by analytic continuation, for

$R(\nu) > -1$, $|\arg p^2| < \pi$, $b>0$.

In particular, when p tends to zero, we get a result [Erdelyi, 1954b, p. 420 (9)]

Some interesting particular cases are given below :—

(i) On taking $\rho = -\nu$, we have

$$\int_0^\infty t^{\nu+1} (p^2+t^2)^{-\nu-2} J_\nu (bt) S_4 \left(1+\nu/2, \tfrac{1}{2}+\nu/2, \mu/2, -\mu/2 ; \frac{b^2(p^2+t^2)}{16}\right) dt$$

$$= \frac{\sqrt{\pi} b^{2+\nu}}{2^{2\nu+3}} K_\mu^2 (bp/2) , \qquad (4)$$

where $R(\nu) > -1$, $|\arg p^2| < \pi$ and $b>0$.

(ii) If we take $\mu=\frac{1}{2}$, we get integral representation for $K_\nu (x)$

$$\int_0^\infty t^{\nu+1} (p^2+t^2)^{\rho/2-3/4} J_\nu (bt) K_{3/2-\rho}(b\sqrt{(p^2+t^2)}) dt$$

$$= 1/b \left(\frac{p}{\sqrt{2}} \right)^{\rho+\nu-\frac{1}{2}} K_{\rho+\nu-\frac{1}{2}}(\sqrt{2}\, bp), \quad (5)$$

where $R(\nu)>-1$, $|\arg p^2| <\pi$ and $b>0$.

(iii) If we also take $\rho=-\nu-\frac{1}{2}$, we get some another integral representation for $K_\nu (x)$.

$$\int_0^\infty t^{\nu+1} (p^2+t^2)^{-\nu-5/2} J_\nu (bt) S_4 \left(3/4+\nu/2, 5/4+\nu/2, \mu/2, -\mu/2 ;\frac{b^2(p^2+t^2)}{16}\right)dt$$

$$=\frac{\pi b^{\nu+2}}{p.2^{2\nu+7/2}} K_{2\mu} (\sqrt{2}\, bp) \quad (6)$$

where $R(\nu)> -1$, $|\arg p^2| <\pi$ and $b>0$.

Example 11. Taking [Saxena, 1960 p. 402 (11)]

$$h(t) = t^\nu e^{t/2b} K_\mu (t/2b)$$

$$\doteqdot \frac{\Gamma(1+\nu\pm\mu) \sqrt{\pi b}\ p^{\frac{1}{2}-\nu}}{\Gamma(3/2+\nu)} {}_2F_1 \left({\frac{1}{2}+\mu, \frac{1}{2}-\mu \atop 3/2+\nu} ; 1-bp\right)$$

$$= \phi (p) ,$$

where $R(1+\nu\pm\mu)> 0$, $R(p)>0$ and $R(b)>0$.

Therefore [Erdelyi, 1954a, p. 285 (65)]

$$t^{-\nu-1}\ e^{-b/t}\, h(t) = t^{-1}\ e^{t/2b-b/t}\ K_{\mu}\ (t/2b)$$

$$\doteqdot 2p\, K_{\mu}\left(\sqrt{bp}+\sqrt{(bp-1)}\right) K_{\mu}\left(\sqrt{bp}-\sqrt{(bp-1)}\right)$$

$$= \Psi(p),$$

where $R(p)>0$.

Applying (2) to the above correspondences, and replacing b by $\frac{1}{4}b^2$ and p by p^2, we obtain

$$\int_0^{\infty} t^{\nu+1}\,(p^2+t^2)^{-\nu-\frac{1}{2}}\, J_{\nu}\,(bt)\ {}_2F_1\left({\frac{1}{2}+\mu,\ \frac{1}{2}-\mu \atop 3/2+\nu};\ 1-\frac{t^2(p^2+t^2)}{4}\right) dt$$

$$= \frac{\Gamma(3/2+\nu)\, b^{\nu-1}}{\Gamma(1+\nu\pm\mu)\ \sqrt{\pi}\, 2^{\nu-1}}\ K_{\mu}\left[\tfrac{1}{2}\left(bp+\sqrt{b^2p^2-4)}\right)\right] \qquad (7)$$

$$K_{\mu}\left[\tfrac{1}{2}\left(bp-\sqrt{b^2p^2-4)}\right),\right]$$

valid by analytic continuation, for

$R(\nu)>-1$, $R(\nu\pm2\mu+3/2)>0$, $|\arg p^2|<\pi$ and $b>0$.

In particular, when p tends to zero, we arrive at a result [Frdelyi, 1954, p. 420 (9)]

Example III. Taking [Saxena, 1960 p. 402 (11)]

$$h(t) = t^{-\rho}\ e^{\frac{1}{2}b/t}\ K_{\mu}\ (b/2t)$$

$$\doteqdot \frac{\cos\mu\pi\ p^{\rho-\frac{1}{2}}}{\sqrt{\pi b}}\ E\,(3/2-\rho,\ 1/2+\mu,\ 1/2-\mu\ ::\ bp)$$

$$= \phi(p),$$

where $R(p)>0$ and $R(\rho)<\frac{1}{2}$.

Therefore also

$$t^{-\nu-1}\ e^{-b/t}\ h(t) = t^{-\nu-\rho-1}\ e^{-\frac{1}{2}b/t} K_{\mu}\ (b/2t)$$

$$\doteqdot \sqrt{\pi}\, p^{\rho+\nu+1}\ G^{30}_{13}\left[pb\,\middle|\,{\frac{1}{2} \atop -\nu-\rho,\ \mu,\ -\mu}\right]$$

$$= \Psi(p),$$

where $R(p)>0$ and $R(b)>0$.

Using the above values of $\phi(p)$ and $\Psi(p)$ in (2) and replacing b by $\frac{1}{4}b^2$ and p by p^2, we obtain

$$\int_0^\infty t^{\nu+1}(p^2+t^2)^{\rho-3/2} J_\nu(bt)\, E\left(\tfrac{3}{2}-\rho, \tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu :: \frac{b^2(p^2+t^2)}{4}\right) dt$$

$$= \frac{\pi b^{\nu+1}\, p^{2\nu+2\rho}}{\cos\mu\pi\; 2^{\nu+2}}\, G_{13}^{30}\left(\frac{b^2p^2}{4}\,\middle|\, \begin{matrix}\tfrac{1}{2}\\ -\rho-\nu, \mu, -\mu\end{matrix}\right), \qquad (8)$$

valid by analytic continuation, for

$R(\nu)>-1$, $R(\nu+2\rho)<5/2$, $|\arg p^2|<\pi$ and $b>0$.

Particular Cases :—(i) when $\rho=-\nu$

$$\int_0^\infty t^{\nu+1}(p^2+t^2)^{-\nu-3/2} J_\mu(bt)\, E\left(3/2+\nu, 1/2+\mu, 1/2-\mu :: \frac{b^2(p^2+t^2)}{4}\right) dt$$

$$= \frac{\pi b^{\nu+1}}{\cos\mu\pi\; 2^{\nu+1}} K_\mu^2(bp/2), \qquad (9)$$

where $R(\nu)>-1$, $|\arg p^2|<\pi$ and $b>0$.

(ii) when $\rho=-\nu-\frac{1}{2}$, we get integral representation for $K_\nu(x)$

$$\int_0^\infty t^{\nu+1}(p^2+t^2)^{-\nu-2} J_\nu(bt)\, E\left(2+\nu, \tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu :: \frac{b^2(p^2+t^2)}{4}\right) dt$$

$$= \frac{\pi\, b^{1+\nu}}{2^{1+\nu}\, p^{\frac{1}{2}}\cos\mu\pi} K_{2\mu}(bp), \quad (10)$$

where $R(\nu)>-1$, $|\arg p^2|<\pi$ and $b>0$.

Example IV. Taking [Saxena, 1960, p. 402(11)]

$$h(t) = t^{-\rho}\, e^{b/t}\, K_\mu(b/t)$$

$$\doteqdot \frac{\cos\mu\pi\; p^{\rho-\frac{1}{2}}}{\sqrt{2\pi b}}\, E\left(3/2-\rho, 1/2+\mu, 1/2-\mu :: 2bp\right)$$

$$= \phi(p),$$

where $R(p)>0$ and $R(\rho)<\frac{1}{2}$.

Therefore also

$$t^{-\nu-1}\ e^{-l/t}\quad h(t) = t^{-\nu-\rho-1}\ \mathrm{K}_{\mu}\ (b/t)$$

$$\doteq \frac{1}{b\sqrt{\pi}}(p/2)^{\nu+\rho}\ \mathrm{S}_4\left(\frac{1-\nu-\rho}{2}, \frac{-\nu-\rho}{2}, \mu/2, -\mu/2\ ;\ bp/4\right)$$

$$= \Psi(p)\ ,$$

where $R(p)>0$ and $R(b)>0$.

Applying (2) to the above correspondences and replacing b by $\frac{1}{4}b^2$ and p by p^2, we obtain

$$\int_0^\infty t^{\nu+1}\ (p^2+t^2)^{\rho-3/2}\ \mathrm{J}_\nu\ (bt)\ \mathrm{E}\left(\frac{3}{2}-\rho, \frac{1}{2}+\mu, \frac{1}{2}-\mu\ ::\ \frac{b^2(p^2+t^2)}{2}\right) dt$$

$$= \frac{b^{\nu-1}\ p^{2(\nu+\rho-1)}}{\cos\mu\pi\ 2^{2\nu+\rho-\frac{1}{2}}}\ \mathrm{S}_4\left(\frac{1-\nu-\rho}{2}, \frac{-\nu-\rho}{2}, \mu/2, -\mu/2\ ;\ b^2p^2/16\right) \tag{11}$$

valid by analytic continuation, for

$R(\nu) > -1$, $|\text{ are } p^2| < \pi$, $R(\nu+2\rho) < 5/2$ and $b>0$.

In particular, when p tends to zero, we get a result [Erdelyi, 1954*b*, p. 420(9)]

Example V. Taking [Saxena, 1960, p. 402(11)]

$$h(t) = t^{-\rho}\ e^{\frac{1}{2}b/t}\ \mathrm{W}_{k,m}\ (b/t)$$

$$\doteq \frac{p^{\rho+k}\ b^k}{\Gamma(\frac{1}{2}-k\pm m)}\ \mathrm{E}\left(1-\rho-k, \tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m\ ::\ bp\right)$$

$$= \phi(p)\ ,$$

where $R(p)>0$ and $R(1-\rho+k)>0$.

Therefore also

$$t^{-\nu-1}\ e^{-b/t}\ h(t) = t^{-\nu-\rho-1}\ e^{-\frac{1}{2}b/t}\ \mathrm{W}_{k,m}\ (b/t)$$

$$\doteq b^{\frac{1}{2}}\ p^{\rho+\nu+3/2}\ \mathrm{G}_{13}^{30}\left(bp\middle|{}^{\ \frac{1}{2}-k}_{-\rho-\nu-\frac{1}{2}, m, -m}\right)$$

$$= \Psi(p),$$

where $R(p)>0$.

Applying (2) to the above correspondences and replacing b by $\frac{1}{2}b^2$ and p by p^2, we obtain

$$\int_0^\infty t^{\nu+1}(p^2+t^2)^{\rho+k-1} J_\nu(bt)\, E\left(1-\rho-k, \tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m :: \frac{b^2(p^2+t^2)}{4}\right) dt$$

$$= \frac{\Gamma(\frac{1}{2}-k\pm m)\, b^{\nu+1-2k}\, p^{2\nu+2\rho+1}}{2^{\nu+2-2k}} G_{13}^{30}\left(b^2p^2/4 \,\middle|\, \begin{matrix} \frac{1}{2}-k \\ -\rho-\nu-\frac{1}{2}, m, -m \end{matrix}\right) \quad (12)$$

valid by analytic continuation, for

$R(\nu)>-1$, $R(\nu+2\rho+2k)<5/2$, $|\arg p^2|<\pi$ and $b>0$.

In particular, when p tends to zero, we get a result [Erdelyi, 1954b, p. 420(9)]

Some interesting particular cases are given below :—

(i) On taking $k=0$ and replacing ρ by $\rho-\frac{1}{2}$, we fall back upon (8).

(ii) If we also replace ρ by $k-\nu-1$, we get another integral representation for $K\ (x)$

$$\int_0^\infty t^{\nu+1}(p^2+t^2)^{2k-\nu-2} J_\nu(bt)\, E\left(2-2k+\nu, \tfrac{1}{2}-k\pm m :: \frac{b^2(p^2+t^2)}{4}\right) dt$$

$$= \frac{\Gamma(\frac{1}{2}-k\pm m)}{p^{1-2k}} (b/2)^{\nu+1-2k} K_{2m}(bp) \quad (13)$$

which on taking $k=0$, gives again (10)

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REFERENCES

1. Erdelyi, A., Tables of Integral Transforms, Vol. I, McGraw Hill, New York, (1954a).

2. ——————, Tables of Integral Transforms, Vol. II, McGraw Hill, New York, (1954b).

3. Gokhroo, D. C., On some Hankel transforms (1964). *Collectanea Mathematica* (in press).

4. Saxena, R. K., Some theorems on generalized Laplace transforms I, *Proc. Nat Inst. Sci. India*, Vol. 26, A. No, 4, p. 400-13, (1960).

ASYMPTOTIC BEHAVIOR OF DIFFERENTIAL SYSTEMS

By

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The behavior of the solutions of $z' = Ax + R(t)x$, where A is a constant or a periodic matrix and R (t), an integrable matrix, has been studied. The results are extended to Differential Equations of order n.

Consider the differential equation $x' = Ax + R(t)x$, where A is a constant matrix and $R(t)$ an integrable matrix. The behavior of solutions of (1) as $t \to \infty$ has been studied in this paper. This method has been applied in a case when the costant matrix A is replaced by a periodic matrix $A(t)$. This result is extended to differential equations of order n when the coefficients a_j are constants and when $a_j(t)$ are periodic functions. $(J = 1, 2, \ldots\ldots\ldots n)$.

Theorem 1:—If

(2) (*a*) The canonical form of A has submatrices J_k, $k \geqq 1$

(*b*) $r + 1$ is the maximum number of rows in any matrix J_k, $k \geqq 1$, $r \geqq 1$

(*c*) $\int_1^\infty t^r \,|\, R(t) \,|\, dt < \infty$

(*d*) λ_j is a characteristic root of A.

(*e*) a fundamental set for $Y' = Ay$ is the n functions $t^k e^{\lambda_j t}$

Then a solution $\varphi(t)$ of (1) is such that

$$e^{-\lambda_j t} \cdot t^{-k} \cdot \varphi(t) \to c \text{ as } t \mapsto \infty$$

where c is a constant vector.

Proof of theorem 1 :—Let $R\lambda_j = \sigma$ where R denotes the real part. The elements of $e^{A(t-s)}$ are divided into two classes. That is

$$(3) \quad e^{A(t-s)} = \gamma_1(t, s) + \gamma_2(t, s)$$

where

$$|\, \gamma_1(t, s) \,| \leqq k_1 \, e^{\sigma(t-s)} \, t^{k-1} \, s^{r-k+1} \qquad (t \geqq s \geqq 1)$$

$$(4) \quad |\, \gamma_2(t, s) \,| \leqq k_2 \, e^{\sigma(t-s)} \, t^k \cdot s^{r-k} \qquad (s \geqq t \geqq 1)$$

k_1, k_2 being constants.

Polynomials multiplying an exponential term in the elements of e^{At} is of degree not higher than r. Now let

(5) $\Psi_o(t) = e^{\lambda_j t} . t^k e + 0 \left(e^{\lambda_j t} . t^{k-1} \right),$

(6) $| \Psi_o(t) | \leqq k_o . e^{\sigma t} . t^k$. and

(7) $$\Psi_{l+1}(t) = \Psi_o(t) + \int_a^t \gamma_1(t, s) R(s) \Psi_l(s) ds - \int_t^{\infty} \gamma_2(t, s) R(s) \Psi_l(s) ds).$$

That is

$$| \Psi_1(t) - \Psi_o(t) | \leqq \int_a^t | \gamma_1(t, s) \| \Psi_o(s) \| R(s) | ds + \int_t^{\infty} | \gamma_2(t, s) \| \Psi_o(s) \| R(s) | ds$$

using the inequetities in (4) and (6)

(8) $$| \Psi_1(t) - \Psi_o(t) | \leqq k_o k_1 \int_a^t e^{\sigma(t - s)} t^{k-1} s^{\delta - k + 1} . e^{\sigma s} s^k | R(s) | ds + k_o k_2 \int_t^{\infty} e^{\sigma(t - s)} . t^k . s^{\delta - k} e^{\sigma s} s^k | R(s) | ds$$

Simplifying the two integrals on the right hand side of (8) and using $t \geqq s$ in the first integral we have

(9) $$| \Psi_1(t) - \Psi_o(t) | \leqq k_o k_1 e^{\sigma t} t^k \int_a^t s^r | R(s) | ds + k_o k_2 e^{\sigma t} t^k \int_t^{\infty} s^r | R(s) | ds$$

$$\leqq k_o (k_1 + k_2) e^{\sigma t} t^k \int_a^{\infty} s^r | R(s) | ds$$

(when a is taken large enough)

$$\leqq \frac{k_o\, e^{\sigma t}\, t^k}{2}$$

Where $(k_1 + k_2) \int_a^\infty s^r \mid R(s) \mid ds < \frac{1}{2}$

and by induction

$$(10) \qquad \mid \Psi_{l+1}(t) - \Psi_l(t) \mid \leqq \frac{k_o\, e^{\sigma t}\, t^k}{2^{l+1}}$$

So that $\{\Psi_l(t)\}$ converges imformly to a limit function $\varphi(t)$ and

$$(11) \qquad \mid \varphi(t) \mid \leqq 2\, k_o\, e^{\sigma t}\, t^k$$

$$(12) \qquad \varphi(t) = \Psi_o(t) + \int_a^t \gamma_1(t, s)\, R(s)\, \varphi(s)\, ds$$

$$\int_t^\infty \gamma_2(t, s)\, R(s)\, \varphi(s)\, ds.$$

where $\Psi_o(t)$ is defined in (5).

$$(13) \qquad \mid \varphi(t) - e^{\lambda j t}\, t^k\, c \mid \leqq \mid O\left(e^{\lambda j t}\, t^{k-1}\right) \mid$$

$$+ \int_a^t \mid \gamma_1(t, s) \| \varphi(s) \| R(s) \mid ds$$

$$+ \int_t^\infty \mid \gamma_2(t, s) \| \varphi(s) \| R(s) \mid ds.$$

using (4) and (11) we obtain

$$(14) \qquad \mid \varphi(t) - e^{\lambda j t}\, t^k\, c \mid \leqq \mid O\left(e^{\lambda j t}\, t^{k-1}\right) \mid$$

$$+ 2\, k_o\, k_1 \int_a^t e^{\sigma(t-s)}\, t^{k-1}\, S^{r-k+1}\, e^{\sigma s}\, S^k \mid R(s) \mid ds$$

$$+ 2 k_c k_2 \int_t^{\infty} e^{\sigma(t-s)} \; t^k s^{r-k} \; e^{\sigma s} \; s^k \; | R(s) | \, ds$$

(15) $$e^{-\sigma t} \; t^{-k} \left| \varphi(t) - e^{\lambda_j t} t^k c \right|$$

$$\leqq e^{-\sigma t} \; t^{-k} \; \left| O\left(e^{\lambda_j t} \; t^{k-1}\right) \right|$$

$$+ 2 k_o k_1 t^{-1} \int_a^t s^{r+1} \; | R(s) | \; ds$$

$$+ 2 k. k_2 \int_t^{\infty} s^r \; | R(s) | \; ds.$$

$$\leqq e^{-\sigma t} t^{-k} \; \left| O\left(e^{\lambda_j t} \; t^{k-1}\right) \right|$$

$$+ 2 k_o k_1 t^{-\frac{1}{2}} \int_a^{t^{\frac{1}{2}}} S^r \; | R(s) | \, ds$$

$$+ 2 k_o (k_1 + k_2) \int_{t^{\frac{1}{2}}}^{\infty} S^r \; | R(s) | \, ds$$

In the first integral on the right hand side $t^{\frac{1}{2}} \geqq S^{\frac{1}{2}}$.

Hence from the above assumptions as $t \to \infty$, the theorem follows.

Theorem 2 :—Consider the n^{th} order differential equations

(16) $$L_n x = x^n + (a_1 + r_1 (t) x^{n-1} + \ldots\ldots + \ldots\ldots\ldots + (a_n + r_n (t)) x = 0$$

and

(17) $$L_n x = x^n + a_1 x^{n-1} + \ldots\ldots\ldots + a_n x = 0$$

If

(18) (*a*) $\lambda_1, \lambda_2, \ldots \ldots, \lambda_s$ be the distinct characteristic roots of

$$f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots\ldots\ldots\ldots + \alpha_n \lambda = 0$$

(b) λ_i has multiplicity m_i $i = 1, 2, \ldots\ldots\ldots\ldots, s$

(c) a fundamental set for (17) is the n functions

$$t^k e^{\lambda_i t} \qquad \begin{pmatrix} k = 0, 1, \ldots\ldots m_i - 1 \\ i = 1, 2, \ldots\ldots\ldots\ldots, s \end{pmatrix}$$

(d) $$\int_1^\infty t^k \left| r_{l+1}(t) \right| dt < \infty$$

$$(l = 0, 1, 2, \ldots\ldots\ldots n - 1)$$

Then a solution $\varphi(t)$ of (14) is such that

$$\lim_{t \to \infty} \left(\varphi^{(m)} - \lambda_i^m e^{\lambda_i t} t^k \right) = 0$$

(m) denotes the m^{th} derivative.

Proof of theorem 2 :—In (16) let $x = x_1$; $x_1' = x_2$;...
$x'_n = -(a_1 + r_1(t)) x_n \ldots\ldots\ldots\ldots\ldots\ldots (a_1 + r_1(t)) x_1$ writing (19) in the vector-matrix from

(20) $\hat{x}' = A\hat{x} + R(t)\hat{x}$ where $\hat{x}$ is a single column vector of n rows.

$$A = \begin{Bmatrix} 0 & 1 & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 1 & 0 \ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ -a_n \ldots\ldots\ldots\ldots & & & -a_1 \end{Bmatrix}$$

$$R(t) = \begin{Bmatrix} 0 & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ \ldots & & \ldots\ldots\ldots\ldots\ldots \\ & & \ldots\ldots\ldots\ldots\ldots \\ -r_n & -r_{n-1} \ldots\ldots\ldots & -r_1 \end{Bmatrix}$$

Since equation (20) is in the form of (1), Theoram 2 can easily be proved from theorem 1.

Theorem 3 :—In the equation

(21) $x^1 = A(t)x + R(t)x$

if

(22) (a) $A(t)$ is a periodic matrix with period ω.

(b) The canonical form of the constant matrix B associated with the periodic matrix $A(t)$ has sub-matrices J_k, $K \geqq 1$ and $r+1$ is the maximum number of rows in any matrix J_k, $K \geqq 1$

(c) λ_j is a multiplier associated with $A(t)$

(*d*) $\int_1^{\infty} t^{\gamma} \quad | R(t) | \, dt < \infty$

(*e*) $Y^1 = A(t) Y$ have a solution of the form

$$e^{\lambda_j t} \; t^k \; p_j \; (t) + O\left(e^{\lambda_j t} t^{k-1} \; p_j(t) \right)$$

where p_j is a periodic column vector with period ω,

then (21) has a solution $\varphi(t)$ such that

$$\lim_{t \to \infty} e^{-\lambda_j t} \, . \, t^{-k} \, . \, \varphi(t) \, p_j(t).$$

Proof of Theorem 3 :—Let $Y^1 = A(t) Y$ has a solution (fundamental) of the from $P(t) \, e^{ct} \, . \, t^k$ where c is a constract matrix and $P(t)$ is a non-singular periodic matrix with period ω.

Substituting this in $Y^1 = A(t) Y$

(23) $P^1 + K P t^{-1} + P C = A P$

$$\text{Let } x = P(t) \, t^k . Z$$

(21) becomes

(24) $Z^1 = C Z + P^{-1} R P Z$

where C is a constant matrix.

(24) is similar to (1). Hence applying theorem 1, the result stated in theorem 3 is obtained

Theorem 4 :—In the equations

(25) $L_n x = x^n + (a_1(t) + r_1(t)) \, x^{n-1} + \ldots\ldots\ldots$

$$+ \ldots\ldots\ldots\ldots\ldots + \left(a_n(t) + r_n(t) \, x = 0 \right.$$

and

(26) $L_n x = x^n + a_1(t) \, x^{n-1} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$\ldots\ldots\ldots\ldots\ldots + a_n(t) \, x = 0$$

if

(27) (*a*) $a_i(t)$ are periodic of period ω $i = 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots n$

(*b*) $\lambda_1 \lambda_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \lambda_s$ are the distinct characteristic exponents

associated with $a_j\,(t)$

(c) λ_i has multiplicity m_i

$$(i = 1, 2, \ldots\ldots\ldots\ldots, s)$$

$$(d) \quad \int_1^{\infty} t^k \left| r_{l+1}(t) \right| dt < \infty$$

$$\left(\begin{array}{l} k = 0, 1, \ldots\ldots\ldots\ldots m_i - 1 \\ l = 0, 1, \ldots\ldots\ldots\ldots\ldots n - 1 \end{array} \right)$$

(e) The fundamental set for (26) is

$$t^k \,.\, e^{\lambda_i t} p_i\,(t)$$

where $p_i\,(t)$ is a periodic column vector with period ω.

Then a solution $\varphi\,(t)$ of (25) is such that

$$(28) \qquad \underset{t \to \infty}{\text{Limit}} \left(\psi^{(m)} - \lambda_j{}^m\, t^k\, e^{\lambda_i t}\, p_i(t) \right) = 0$$

$$\left\{ \begin{array}{l} i = 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots s \\ k = 0, 1, \ldots\ldots\ldots\ldots\ldots\ldots m_i - 1 \\ m = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots n - 1 \end{array} \right\}$$

(m) denotes the m^{th} derivative.

Proof of theorem 4 :—As in theorem 2, equation (25) can be written in the vector-matrix form

$$(29) \qquad \hat{x}^1 = A\,(t)\,\hat{x} + R\,(t)\,\hat{x}$$

where

$$A\,(t) = \left\{ \begin{array}{cccc} 0 & 1 & 0 & 0\ldots\ldots\ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 1 & 0\ldots\ldots\ldots\ldots\ldots\ldots 0 \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ -a_n & \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots -a_1} \end{array} \right\}$$

and

$$R\,(t) = \left\{ \begin{array}{cccc} 0 & 0 & 0 & \ldots\ldots\ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 0 & \ldots\ldots\ldots\ldots\ldots\ldots 0 \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ -r_n & -r_{n-1} & \multicolumn{2}{c}{\ldots\ldots\ldots\ldots\ldots -r_1} \end{array} \right\}$$

(29) is in the form of (21). Applying the above theorems, the result follows.

Remark :—As $t \to \infty$, the solutions of the equations in the above theorems behave in a manner as if $R(t)$ is $\equiv 0$.

REFERENCE

1. E. A. Coddington and N. Levension : "Theory of Ordinary Differential Equations" Mcgraw-Hell & Co., New York, 1955.

UNIVERCITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

By

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ABSTRACT

Considerable research has been done on the problem of uniqueness of solutions of the system

$$x' = f(t, x)$$

The method of successive approximations has been used to prove the theorems of this paperi. A lemma which has been used also where by the author has been used here to prove theorem 3 of this paper.

In this paper we discuss the existence and uniqueness of solutions of the differential system

(1) $$x' = f(t, x)$$

Considerable research has been done on the problem of uniqueness of solutions of the system (1). The method of successive approximations has been used to prove the theorems of this paper.

Theorem I: – If

(2) (a) $$|f(t, \xi)| \leqq k(t)\ (1 + |\xi|)$$

(b) $$|f(t, x) - f(t, \bar{x})| \leqq k(t)\ |x - \bar{x}|$$

for (t, x) and $(t, \bar{x})$ in R where R is the rectangle defined by

R: $|t - \tau| \leqq a \quad |x - \xi| \leqq b.$

where (τ, ξ) is some point in the (t, x) please and $a > 0$, $b > 0$

(c) $f(t, \psi(t))$ is integrable for any continuous function Ψ, then there exists an unique solution of (1) to which the successive approximations converge uniformly in the interval $[\tau, \tau + \alpha_1]$.

Proof of theorem I:—The successive approximations of (1) are the functions $\theta_0, \theta_1\ \theta_2 \ldots\ldots$ given recussively as $\theta_0(t) = \xi$

(3) $$\theta_{k+1}(t) = \xi + \int_{\tau}^{t} f(s, \theta_k(s))\, ds$$

$$(k = 0, 1, 2, \ldots\ldots\ |t - \tau| \leqq \alpha_1)$$

that is $|\theta_1(t) - \xi| \leqq \int_{\tau}^{t} f(s, \theta_0(s))\, ds$

using 2 (a) we have

$$(4) \qquad |\theta_1(t) - \xi| \leqq (1 + |\xi|) \int_\tau^t K(s)\, ds$$

$$\leqq (1 + |\xi|) K(t)$$

where $K(t) = \int_\tau^t k(s)\, ds$

and it follows by induction

$$(5) \qquad |\theta_n(t) - \theta_{n-1}(t)| \leqq (1 + |\xi|) \frac{(K(t))^n}{n!}$$

If will be shown that every θ_k exists on $[\tau, \tau + \alpha_1]$. $\theta_k \ \varepsilon \ c^1$

$$(6) \text{ and } \quad |\theta_k(t) - \xi| \leqq (1 + |\zeta|)(e^{K(t)} - 1)$$

$$[t \ \varepsilon \ [\tau, \tau + \alpha_1]$$

Since $\theta_0(t) = \xi$ equation (6) is satisfied

$$(7) \quad |\theta_k(t) - \xi| \ | \leqq \theta_k(t) - \theta_{k-1}(t)| + |\theta_{k-1}(t) - \theta_{k-2}(t)|$$

$$+ \ldots\ldots\ldots\ldots + |\theta_1(t) - \xi|$$

Using (5) we can easily deduce

$$(8) \qquad |\theta_k(t) - \xi| \leqq (1 + |\xi|)(e^{K(t)} - 1).$$

This means geometrically that all the θ_k start at (τ, ξ) and stay with in a region

$$|x - \xi| \leqq (1 + (\xi))\, e^{k(t)} - 1)$$

for $t \ \varepsilon \ [\tau, \tau + \alpha_1]$

From (5) and 6) it follows that the sequence $[\theta_n]$ is uniformly convergent and tends uniformly to a continuous limit function θ which is a solution of (1) on $[\tau, \tau + \alpha_1]$ for which $\theta(\tau) = \xi$.

Theorem 2:—Let $f \ \varepsilon \ (c, lip)$ in D and $(\tau, \xi) \ \varepsilon$ D. If θ_1 and θ_2 are any two solutions of (1) on (a, b) $a < \tau < b$ such that

$\theta_1(\tau) = \theta_2(\tau) = \xi$ then $\theta_1 = \theta_2$

Proof of theorem 2:—The proof is given in [1].

By this theorem, θ in the proof of theorem 1 is unique.

Remark:—If f in the above theorem is defined for all x and $\tilde{x}$ and for all $t \in [a, b]$ and if $\tau \in [a, b]$ then the successive approximations converge uniformly on on $[a, b]$.

The following Lemma is used in the proof of theorem 3.

Lemma:—*If*

(9) $$\theta(t) \leqq \Psi(t) + \int_a^t \chi(s)\,\theta(s)\,ds$$

where θ, Ψ, χ be real-valued continuous (or pieceivese continuous) functions on a real t interval $I : a \leqq t \leqq b$ and $\chi(t) \geqq 0$ on I then

(10) $$\theta(t) \leqq \Psi(t) + \int_a^t \chi(s)\,\Psi(s) \exp\left(\int_a^t \chi(u)\,du\right) ds$$

Proof of the Lemma:—This Lemma has been by the author in [2].

Theorem 3:—In the initial values problem

(11) $$x' = A(t)\,x + b(t);\; x(\tau) = \xi$$

if $\tau \in [a, b]$ and if the matrix $A(t)$ and the vector b be integrable functions of t over $[a, b]$ and if

$$|A(t)| \leqq k(t);\quad |b(t)| \leqq k(t)$$

where $\int_a^b K(t)\,dt < \infty$

Then there is a unique solution θ over $[a, b]$ in the sense that $\theta \in c$ and

(13) $$\theta(t) = \xi + \int_\tau^t A(s)\,\theta(s)\,ds + \int_\tau^t b(s)\,ds \text{ on } [a, b].$$

Proof of Theoem 3:— Let $\theta_0(t) = \xi$

(14) and $$\theta_{j+1}(t) = \xi + \int_\tau^t A(s)\,\theta_j(s)\,ds + \int_\tau^t b(s)\,ds$$

(15) $$|\theta_j(t) - \theta_{j-1}(t)| \leqq \int_T^t |A(s)|\, |\theta_{j-1}(s) - \theta_{j-2}(s)|\, ds$$

using the inequalities in (12), (15) yields

(16) $$|\theta_1(t) - \xi| \leqq \int_T^t |A(s)|\, |\xi|\, ds + \int_T^t |b(s)|\, ds$$

$$\leqq (1 + |\xi|)\, K(t)$$

where $K(t) = \int_T^t K(s)\, ds.$

and similarly

(17) $$|\theta_n(t) - \theta_{n-1}(t)| \leqq (1 + |\xi|) \frac{(K(t))^n}{n!}$$

Hence a_j converges uniformly over $[a, b]$. If this holds for all $b < \bar{b}$ then the solution exists over $[a, \bar{b})$, where $\bar{b}$ may be taken as ∞. A similar case occurs at the left end point. Let $\theta_j(t)$ converges to $\theta(t)$. Then

(18) $$\theta(t) = \xi + \int_T^t A(s)\,\theta(s)\, ds + \int_T^t b(s)\, ds$$

Now it will be shown that $\theta(t)$ is unique.

Let $\Psi(t)$ be another solution of (1)

Then we have

(19) $$\Psi(t) = \xi + \int_T^t A(s)\,\Psi(s)\, ds + \int_T^t b(s)\, ds$$

Hence $|\theta(t) - \Psi(t)| \leqq 0 + \int_T^t |A(s)|\, |\theta(s) - \Psi(s)|\, ds$

That is

(20) $$p(t) \leqq 0 + \int_T^t K(s)\, p(s)\, ds$$

where $p(t) = |\theta(t) - \Psi(t)|$

Applying the Lemma (above), we get the result of Theorem 3.

Here the convergence of the successive approximations imply uniqueness holds good, a through it is not always applicable.

REFERENCES

1. Coddington and Levension:—"Theory of ordinary Differential Equations" Mcgraw Hill & Co., New York, 1955.

2. D. Ramakrishna Rao:—"Unicity of solutions of a differential Equation" Monatschefte Fiir Mathematik (To appear).

CERTAIN RELATIONSHIPS BETWEEN VARIOUS TRANSFORMS

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ABSTRACT

The purpose of this note is to study a few relationships existing between Laplace, Meijer, Stieltjes and Hankel transforms in the form of theorems and sequences. Most of the results given earlier by Saxena (1959) follow as their particular cases. An infinite integral has also been evaluated with the help of the theorem.

1. Introduction:—A function $\phi(p)$ is called the generalized Stieltjes transform of $h(t)$ of order λ, when they are connected by the relation

$$\phi(p) = p \int_0^\infty (p+t)^{-\lambda}\, h(t)\, dt, \tag{1·1}$$

provided the integral is convergent and $|\arg p| < \pi$.

The object of this paper is to obtain certain relations between Laplace, Meijer, and generalized Stieltjes transform with that of Hankel transform. The results proved are quite general and include as particular cases, certain theorems given earlier by Saxena (1959).

The transforms which will be required in our investigation later on are:

(i) the Laplace transform defined by

$$\phi(p) = p \int_0^\infty e^{-pt}\, h(t)\, dt, \tag{1.2}$$

(ii) the Meijer transform by

$$\phi(p) = \sqrt{\frac{2}{\pi}}\, p \int_0^\infty (pt)^{\frac{1}{2}}\, K_\nu(pt)\, h(t)\, dt, \tag{1.3}$$

and (iii) the Hankel transform by

$$\phi(p) = p \int_0^\infty (pt)^{\frac{1}{2}}\, J_\nu(pt)\, h(t)\, dt \tag{1.4}$$

(1.3) reduces to (1.2) when $\nu = \pm\frac{1}{2}$ on account of the well known identity

$$K_{\pm\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \tag{1.5}$$

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Throughout this paper, we shall represent (1.1), (1.2), (1.3) and (1.4) respectively by

$$\phi(p) \overset{S}{\underset{\lambda}{=}} h(t),\ \phi(p) \doteq h(t),$$

$$\phi(p) \overset{K}{\underset{\nu}{=}} h(t) \text{ and } \phi(p) \overset{J}{\underset{\nu}{=}} h(t) \tag{1.6}$$

2. The following results will be required in the sequel.

$$x^{\sigma-1}\, G^{\alpha\beta}_{\gamma\delta}\left(zx^{n/s} \,\middle|\, \begin{matrix} a_1 \ldots\ldots a_\gamma \\ b_1 \ldots\ldots b_\delta \end{matrix} \right) \overset{S}{\underset{\lambda}{=}} (2\pi)^{(1-n)+(1-s)(\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)}$$

$$n^{\lambda-1}\, s^{\Sigma b_\delta - \Sigma a_\gamma + \gamma/2-\delta/2+1} \times$$

$$p^{1+\sigma-\lambda}\, G^{s\alpha+n,\ s\beta+n}_{s\gamma+n,\ s\delta+n}\left(\frac{p^n z^s}{s^{s(\delta-\gamma)}} \,\middle|\, \begin{matrix} \Delta(s;a_1)\ldots\Delta(s;a_\beta),\ \Delta(n;1-\sigma),\ \Delta(s;a_{\beta+1})\ \ldots\ \Delta(s;a_\gamma) \\ \Delta(s;b_1)\ldots\Delta(s;b_\alpha),\ \Delta(n;\lambda-\sigma),\ \Delta(s;b_{\alpha+1})\ \ldots\ \Delta(s;b_\delta) \end{matrix} \right) \tag{2.1}$$

where n and s are +ve integers, and $\alpha+\beta > \frac{1}{2}(\gamma+\delta)$,

$$|\arg z| < (\alpha+\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta)\pi,\quad |\arg p| < \pi,\ 0 \leqslant \alpha \leqslant \delta,\ 0 \leqslant \beta \leqslant \gamma,$$

$$R\left[n/s \min b_j \right] > R(-\sigma) > R\left[n/s\ a_f - \lambda - n/s \right]$$

$$\left[j=1,2\ldots\alpha \right] \left[f=1,2,\ldots\beta \right]$$

and the symbol $\Delta(n;\alpha)$ denote the set of papameters

$$\frac{\alpha}{n},\ \frac{\alpha+1}{n},\ \frac{\alpha\ \ 2}{n} \ldots\ldots \frac{\alpha+n-1}{n} \text{ respectively.}$$

This follows from Saxena's formula [1959, p. 341 (10)] on using the property of the G-Function due to Saxena [1960 p. 401 (7)]

Now we below give some interesting particular cases of (2.1) obtained by giving suitable values to its parameters.

In what follows n and s are +ve integers, and

$$t^{\sigma-1} K_\nu\left(at^{1/s} \right) \overset{S}{\underset{\lambda}{=}} \frac{(2\pi)^{(1-s-2n)}}{\Gamma(\lambda)} (2n)^{\lambda-1} p^{1+\sigma-\lambda}$$

$$G^{2s+2n,\ 2n}_{2n,\ 2s+2n}\left(\frac{a^{2s}\ p^{2n}}{(2s)^{2s}}\middle| \begin{matrix}\Delta\ (2n;\ 1-\sigma)\\ \Delta\ (s;\pm\nu/2),\ \Delta\ (2n;\ \lambda-\sigma)\end{matrix}\right),$$

where $R\ (\sigma - n/s\ \nu) > 0$, $|\arg a| < \pi$ (2.2)

$$x^{\sigma-1}\ e^{-\frac{1}{2}ax^{n/s}}\ W_{k,\ m}\ (ax^{n/s})\ \underset{\lambda}{\overset{S}{=}}\ \frac{(2\pi)^{3/2\ -\ s/2-n}}{\Gamma\ (\lambda)}\ n^{\lambda-1}\ s^{k+\frac{1}{2}}\ p^{1+\sigma-\lambda}\ \times$$

$$G^{2s+n,\ n}_{s+n,\ 2s+n}\left(\frac{a^{s}\ p^{n}}{s^{s}}\middle| \begin{matrix}\Delta\ (n;\ 1-\sigma),\ \Delta\ (s;\ 1-k)\\ \Delta\ (s;\ \tfrac{1}{2}\pm m),\ \Delta\ (n;\ \lambda-\sigma)\end{matrix}\right),$$

$R\ (p) > 0$, $R\ [\sigma + n/s\ (\frac{1}{2}\pm m)] > 0$ and $|\arg a| < \pi/2$ (2.3)

In addition to these, the following results [5, 1964b] and [4, 1964*a*] will also be required.

$$p^{\sigma}\ (a+p^{n/s})^{-\lambda}\ \underset{\nu}{\overset{K}{=}}\ \frac{(2s\pi)^{n-2s+\frac{1}{2}}}{\Gamma\ (\lambda)\ a^{\lambda}}\ (2n)^{\sigma-\frac{1}{2}}\ (2s)^{\lambda}\ t^{-\sigma}\ \times$$

$$G^{2s,\ 2s}_{2s,\ 2s+2n}\left(\frac{a^{2s}\ t^{2n}}{(2n)^{2n}}\middle| \begin{matrix}\Delta\ (2s;\ 1)\\ \Delta\ (2s;\ \lambda),\ \Delta\left(n;\ \dfrac{2\sigma\pm2\nu+1}{4}\right)\end{matrix}\right), \tag{2.4}$$

where $R\ (p) > 0$, $2s > n$, $|\arg a^{s}| < (2s-n)\ \pi$,

$R\ (n\lambda \pm s\nu - s\sigma + 3/2s) > 0$.

$$p\ (a+t^{n/s})^{-\lambda}\ \underset{\nu}{\overset{K}{=}}\ \frac{(2\pi)^{3/2-n-2s}}{\Gamma\ (\lambda)\ a^{\lambda}}\ (2n)^{\rho+\frac{1}{2}}\ (2s)^{\lambda}\ p^{-\rho}\ \times$$

$$G^{2s+2n,\ 2s}_{2s,\ 2s+2n}\left(\frac{a^{2s}\ p^{2n}}{(2n)^{2n}}\middle| \begin{matrix}\Delta\ (2s;\ 1)\\ \Delta\left(n;\ \dfrac{3+2\rho\pm2\nu}{4}\right),\ \Delta\ (2s;\ \lambda)\end{matrix}\right) \tag{2.5}$$

where $|\arg a| < \pi$, $R\ (p) > 0$, $R\ (\rho \pm \nu + 3/2) > 0$.

3. *Theorem 1.* If

$$\phi\ (p) \doteq h\ (t)\ , \tag{3.1}$$

and $$\Psi\ (p)\ \underset{\lambda}{\overset{s}{=}}\ t^{-s/n\ \rho-1}\ e^{-tt^{s/n}}\ \phi\ \left(t^{s/n}\right),$$

then

$$\Psi\ (p) = \frac{p^{1-\lambda}\ (2\pi)^{3/2-n/2-s}\ n^{3/2-\rho}}{\Gamma\ (\lambda)\ s^{1-\lambda}}\int_0^{\infty}(t+b)^{\rho-1}$$

$$G^{s+n,\, s}_{s,\, s+n}\left(\frac{p^{s}\,(t+b)^{n}}{n^{n}}\middle|\begin{matrix}\Delta\,(s;\,1)\\ \Delta\,(n;\,1-\rho),\ \Delta\,(s;\,\lambda)\end{matrix}\right) h\,(t)\,dt\ , \tag{3·2}$$

provided that the integral is convergent and the Laplace transform of $|\,h\,(t)\,|$ and the generalized Stieltjes transform of $|\,t^{-s/n\rho-1}\,e^{-b\,t^{s/n}}\,\phi\,(t^{s/n})\,|$ exists, $R\,(p)>0$, $R\,(p+b)>0$ and $R\,(1-\rho)>0$.

Proof:—Taking [Saxena 1960, p. 404 (17)]

$$e^{-bt}\,t^{-\rho}\left(a+t^{n/s}\right)^{-\lambda} \doteq \frac{(2\pi)^{\frac{1}{2}(3-n-2s)}}{\Gamma\,(\lambda)\,a^{s}}\, S^{\lambda}\, n^{\frac{1}{2}-\rho}\ p\,(p+b)^{\rho-1}\ \times$$

$$G^{s+n,\, s}_{s,\, s+n}\left(\frac{a^{s}\,(p+b)^{n}}{n^{n}}\middle|\begin{matrix}\Delta\,(s;\,1)\\ \Delta\,(n;\,1-\rho),\ \Delta\,(s;\,\lambda)\end{matrix}\right), \tag{3·3}$$

where $R\,(p+b)>0$ $|\arg a|<\pi$ $R\,(1-\rho)>0$.

Applying Parseval Goldstein theorem (3, p. 105) of the Operational Calculus to (3.1) and (3.3), we obtain

$$\int_0^\infty t^{-\rho-1}\,e^{-bt}\left(a+t^{n/s}\right)^{-\lambda}\phi\,(t)\,dt = \frac{(2\pi)^{3/2-n/2-s}\,S^{\lambda}\,n^{\frac{1}{2}-\rho}}{\Gamma\,(\lambda)\,a^{\lambda}}$$

$$\int (t+b)^{\rho-1}\, G^{s+n,\, s}_{s,\, s+n}\left(\frac{a^{s}\,(t+b)^{n}}{n^{n}}\middle|\begin{matrix}\Delta\,(s;\,1)\\ \Delta\,(n;\,1-\rho),\ \Delta\,(s;\,\lambda)\end{matrix}\right) h\,(t)\,dt$$

the theorem follows immediately from above on multiplying both sides by a and replacing a by p.

Cor. I. When $s=1$ the theorem takes the following form

If $\quad \phi\,(p) \doteq h\ t)\ ,$

and $\quad \Psi\,(p) \overset{S}{\underset{\lambda}{=}} t^{-\rho/n-1}\,e^{-bt^{1/n}}\,\phi\left(t^{1/n}\right),$

then

$$\Psi\,(p) = \frac{(2\pi)^{\frac{1}{2}(1-n)}\ n^{3/2-\rho}}{\Gamma\,(\lambda)\,p^{\lambda-1}}\int_0^\infty (t+b)^{\rho-1}\,E\left[\Delta\,(n;\,1-\rho),\,\lambda : p\left(\frac{t+b}{n}\right)^{n}\right] h\,(t)\,dt\ , \tag{3.4}$$

provided that the integral is convergent and the Laplace transform of $|\,h(t)\,|$ and the generalized Stieltjes transform of $|\,t^{-\rho/n-1}\, e^{-bt^{1/n}}\, \phi(t^{1/n})$ exist, $R(p+b) > 0$.

Cor. II. If we take $n=s=1$, and $\lambda=\frac{1}{2}-k-\nu$, $\rho=\frac{1}{2}+k-\nu$ the G-Function of (3.2) degenerates into a Whittaker function [Erdelyi 1954b, p. 535,] and we get the theorem given by Saxena [1959, p. 344]

Example 1. Taking from (2.3)

$$t^{-s/n\,\rho-1}\, e^{-bt^{s/n}}\, \phi\left(t^{s/n}\right) = t^{\sigma-1}\, e^{-\frac{1}{2} b\, t^{s/n}}\, W_{l,\,m}\left(b\, t^{s/n}\right)$$

$$\stackrel{S}{\underset{\lambda}{=}} \frac{(2\pi)^{3/2-n/2-s}}{\Gamma(\lambda)}\, S^{\lambda-1}\, n^{l+\frac{1}{2}}\, p^{1+\sigma-\lambda} \times$$

$$G^{s+2n,\,s}_{s+n,\,2n+s}\left(\frac{b^n s^s}{n^n}\,\middle|\,\begin{matrix}\Delta(s;1-\sigma),\,\Delta(n;1-\rho)\\ \Delta(n;\frac{1}{2}\pm m),\,\Delta(s;\lambda-\sigma)\end{matrix}\right) = \psi(p)\ ,$$

where $R\,[n\,\sigma + s\,(\frac{1}{2}\pm m] > 0$, $R\,(p) > 0$.

therefore from Saxena [1961, p. 41 (17)] we have

$$\phi(p) = p^{\rho+n/s\,\sigma}\, e^{\frac{1}{2} bp}\, W_{l,\,m}(bp)$$

$$\doteq \frac{b^e\, t^{-\rho-n/s\,\sigma-l}}{\Gamma(1-l-\rho-n/s\,\sigma)}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}-l-m,\ \frac{1}{2}-l-m\\ 1-l-e-n/s\,\sigma\end{matrix};\ -\frac{t}{b}\right] = h(t)\ ,$$

where $R\,(1-\rho-n/s\sigma) > 0$, $R\,(p)$

Applying theorem I to the above relations, we obtain

$$\int_0^\infty t^{-l-\rho-n/s\,\sigma}\,(t+b)^{\rho-1}\, {}_2F_1\left[\begin{matrix}\frac{1}{2}-l\pm m\\ 1-l-\rho-n/s\,\sigma\end{matrix};\ -\frac{t}{b}\right]$$

$$G^{s,\,s+n}_{s+n,\,s}\left(\frac{n^n}{p^s\,(t+b)^n}\,\middle|\,\begin{matrix}\Delta(n;\rho),\,\Delta(s;1-\lambda)\\ \Delta(s;0)\end{matrix}\right) dt$$

$$= \frac{\Gamma(1-l-\rho-n/s\,\sigma)p^{\sigma}\, n^{l+\rho-1}}{t^l}\, G^{2n+s,\,s}_{s+n,\,2n+s}\left(\frac{a^n\, p^s}{nn}\,\middle|\,\begin{matrix}\Delta(s;1-\sigma),\,\Delta(n;1-l)\\ \Delta(n;\frac{1}{2}\pm m),\,\Delta(s;\lambda-\sigma)\end{matrix}\right), \qquad (3.5)$$

where $R(1-l-\rho-n/s\,\sigma) > o$, $R(\frac{1}{2}+n/s\,\sigma \pm m) > o$, $R(p) > o$, $R(b) > o$.

In particular if we take $n=s=1$, and $\rho=\frac{1}{2}+k-v$, $\lambda=\frac{1}{2}-k-v$, and $\sigma=1+\mu$, we get the result obtained by Saxena [1959, p. 345 (22)]

Theorem II. If $\phi(p) \doteq h(t)$

and $$\Psi(p) \underset{\lambda}{\overset{S}{=}} t^{s/n\,\rho-1} e^{-bt^{s/n}} h\left(t^{s/n}\right),$$

then

$$\Psi(p) = \frac{(2\pi)^{\frac{1}{2}(1+n-2s)}\, n^{\rho+\frac{1}{2}}\, S^{\lambda-1}}{\Gamma(\lambda)\;p^{\lambda-1}} \int_0^\infty t^{-\rho}(t+b)^{-1}$$

$$G^{s\;s}_{s,\,s+n}\left(\frac{p^s\,t^n}{n^n}\middle|\begin{matrix}\Delta(s;1)\\ \Delta(s;\lambda),\ \Delta(n;p)\end{matrix}\right)\phi(t+b)\times dt\ , \tag{3.6}$$

provided that the integral is convergent and the Laplace transform of $|\,h(t)\,|$ and the generalized Stieltjes transform of $|\,t^{s/n\,\rho-1} e^{-bt^{s/n}} h(t^{s/n})\,|$ exist, $R(p+b) > o$, $2s > n$, and $R(s+n\lambda-s\rho) > o$.

Proof: The proof is similar to that of theorem I, if we use [Saxena 1961, p. 41 (18)]

$$p^{\rho}\left(a+p^{n/s}\right) \doteq \frac{(2\pi)^{\frac{1}{2}(1+n-2s)}}{\Gamma(\lambda)\quad a^{\lambda}}\; S^{\lambda}\, n^{\rho-\frac{1}{2}}\, t^{-\rho}$$

$$G^{s,\,s}_{s,\,s+n}\left(\frac{a^s\,t^n}{n_n}\middle|\begin{matrix}\Delta(s;1)\\ \Delta(s;\lambda),\ \Delta(n;\rho)\end{matrix}\right), \tag{3·7}$$

$R(s+n\lambda-s\rho) > o$, $|\arg a^s| < \frac{1}{2}(2s-n)\pi$ and $2s > n$
insted of (3.3)

Cor 1. On putting $n=s=1$, the theorem can be enunciated as follows:

If $$\phi(p) \doteq h(t)\ ,$$

and $$\Psi(p) \underset{\lambda}{\overset{S}{=}} t^{\rho-1} e^{-bt} h(t)\ ,$$

then

$$\Psi(p) = \frac{p}{\Gamma(1+\lambda-\rho)} \int_0^\infty t^{\lambda-\rho} (t+b)^{-1} {}_1F_1 \left(\begin{matrix} \lambda \\ 1+\lambda-\rho \end{matrix} ; -pt \right) \phi(t+b)\, dt , \tag{3.8}$$

provided that the integral is convergent, and the Laplace transform of $| h(t) |$ and the generalized Stieltjes transform of $| t^{\rho-1} e^{-bt} h(t) |$ exist, $R(p+\lambda) > 0$, $R(1+\lambda) > 0$.

Also on taking $\rho = 1$ in (3.8) we arrive at an interesting result, that if

$$\phi(p) \doteq h(t) ,$$

and $\Psi(p) \underset{\lambda}{\overset{S}{=}} e^{-bt} h(t)$,

$$\text{then } \Psi(p) \doteq \frac{1}{\Gamma(\lambda)} \frac{t^{\lambda-1}}{(t+b)} \phi(t+b) , \tag{3.9}$$

provided that $R(\lambda) > 0$ and $R(p+b) > 0$

4. *Theorem III.* If $\phi(p) \underset{v}{\overset{K}{=}} t^\rho f(t^{n/s})$, (4.1)

and $\Psi(p) \underset{\lambda}{\overset{S}{=}} t^{s/n(\rho+\sigma)-1} f(t)$,

then

$$\Psi(p) = \frac{(2\pi)^{n-2s+\frac{1}{2}} (2n)^{\sigma+\frac{1}{2}} (2s)^{\lambda-1}}{\Gamma(\lambda) p^{\lambda-1}} \int_0^\infty t^{\sigma-1} G^{2s,\, 2s}_{2s,\, 2s+2n}$$

$$\left(\frac{p^{2s} t^{2n}}{(2n)^{2n}} \middle| \begin{matrix} \Delta(2s; 1) \\ \Delta(2s; \lambda), \Delta\left(n; \frac{2\sigma \pm 2v+1}{4}\right) \end{matrix} \right) \phi(t)\, dt , \tag{4.2}$$

provided that the integral is convergent and the Meijer transform of $| t^\rho f(t^{n/s}) |$ and the generalized Stieltjes transform of $| t^{s/n(\rho+\sigma)-1} f(t) |$ exist, $R(p) > 0$, $2s > n$, $R(n\lambda \pm sv - s\sigma + 3/2s) > 0$.

Proof: Applying Goldstein theorem in case of Meijer transform [Sharma 62] to (4.1) and (2.4) and changing a to p, we get the theorem stated.

Cor 1. when $\nu = \frac{1}{2}$, the theorem can be stated as follows;

If $$\phi(p) \doteq t^{\rho} f\left(t^{n/s}\right),$$

and $\Psi(p) \underset{\lambda}{\overset{S}{=}} t^{s/n(\rho+\sigma)-1} f(t)$,

then $$\Psi(p) = \frac{(2n)^{n-2s+\frac{1}{2}}(2n)^{\sigma+\frac{1}{2}}}{\Gamma(\lambda)\,p^{\lambda-1}}(2s)^{\lambda-1}\int_0^\infty t^{-1-\sigma} \times$$

$$G^{2s,\,2s}_{2s,\,2s+2n}\left(\frac{p^{2s}\,t^{2n}}{(2n)^{2n}}\middle|\begin{matrix}\Delta(2s;1)\\ \Delta(2s;\lambda),\ \Delta\left(n;\frac{2\sigma\pm1+1}{4}\right)\end{matrix}\right)\phi(t)\,dt, \tag{4.3}$$

provided that the integral is convergent, and the Laplace transform of $|t^{\rho} f(t^{n/s})|$ and the generalized Stieltjes transform of $|t^{s/n(\rho+\sigma)-1} f(t)|$ exist, $R(p) > 0$, $2s > n$, $R(2n\lambda \pm s - 2s\sigma + 3s) > 0$.

Some particular cases of the above corollary are worth mentioning:—

(i) when $n=1$, $s=2$, $\rho=0$ and $\sigma=\frac{1}{2}$, the G-Function of (4.3) degenerates into a parabolic cylinder function [Erdelyi [1954*b*, p 392] and it gives the theorem [Saxena 1959, p. 43].

(ii) On the other hand if $n=s=1$ and $\rho=1-\mu$, $\sigma=\mu$, then the G-Function of (4. 3) breaks up into ${}_1F_1$ by virtue of [Erdelyi 1954*a*, p. 384] and we get another theorem due to Saxena 1959, p. 347]

Also if $\mu=1$, we find that

If $$\phi(p) \doteq h(t),$$

$$\Psi(p) \underset{\lambda}{\overset{s}{=}} h(t),$$

and

then $$\Psi(p) \doteq \frac{t^{\lambda-2}}{\Gamma(\lambda)}\phi(t), \tag{4.4}$$

provided that $R(\lambda) > 0$ and $R(p) > 0$.

(4.4) is a particular case of (3.9) when $b \to \infty$.

(*iii*) lastly if we take $\rho=1+2\mu$, $\sigma=1-2\mu$ and also let n tends to 2 and s tends to 1, the G-Function in (4.3) again breaks up into ${}_1F_2$ by virtue of [Erdelyi 1954*a*, p. 384].

Also on taking $\mu = o$ and $\lambda=\nu+\frac{1}{2}$, we obtain that

If $\qquad \phi(p) \doteq t f(t^2)$,

and $\qquad \Psi(p) \underset{\nu+\frac{1}{2}}{\overset{S}{=}} f(t)$.

then $$p^{\nu-\frac{1}{2}}\, \Psi(p^2) \underset{\nu}{\overset{J}{=}} \frac{\Gamma(\nu+1)\, 2^{\mu+1}}{\Gamma(2\nu+1)}\, t^{\nu-3/2}\, \phi(t) \, , \tag{4.5}$$

provided that $R(2\nu+1) > o$ and the integrals involved converge absolutely and uniformly, $p > o$.

Theorem IV. If $p^{\alpha}\, \phi(p) \underset{\nu}{\overset{K}{=}} f(t)$,

and $\qquad \Psi(p) \underset{\lambda}{\overset{S}{=}} t^{s/n\,(\alpha+\rho)-1}\, \phi\left(t^{s/n}\right)$,

then

$$\Psi(p) = \frac{(2\pi)^{3/2-2s-n}\,(2n)^{\rho+3/2}\,(2s)^{\lambda-1}}{\Gamma(\lambda)}\, p^{1-\lambda} \int_0^{\infty} t^{-\rho-1} \times$$

$$G^{2s+2n,2s}_{2s,\,2s+2n}\left(\frac{p^{2s}\, t^{2n}}{(2n)^{2n}} \,\middle|\, \begin{matrix} \Delta(2s;\,1) \\ \Delta\left(n;\, \frac{3+2\rho\pm 2\nu}{4}\right),\ \Delta(2s;\,\lambda) \end{matrix} \right) f(t)\, dt \, , \tag{4.9}$$

provided that the integral is convergent and the Meijer transform of $|f(t)|$ and the generalized Stieltjes transform of $|\, t^{s/n\,(\alpha+\rho)-1}\, \phi(t^{s/n})\, |$ exist, $R(\rho\pm\nu+3/2) > o$, $R(p) > o$.

Proof:—Apply Goldstein theorem in case of Meijer transform to (4·8) and and (2·5) and changing a to p we arrive at the result stated.

REFERENCES

1. Erdelyi, A., Tables of integral Transforms, Vol, I, McGraw Hill, New York. [1954*a*]
2. Tables of Integral Transforms Vol. II, McGraw Hill, New York. [1954*b*]
3. Goldstein, S., Operational representation of Whittaker's confluent hypergeometric functions and Weber's parabolic cylinder functions. *Proc. London Math. Soc.* (2) 34, p. 103-25. [1932]
4. Gupta, K. C, Some theorems on Laplace, Meijer and Varma transforms. *Proc. Nat. Acad. Sci. India*, Vol 34, part II, p. 145-194 [194*a*]
5. On the Inverse Meijer transform of G-Function Collectanea Mathematica, Barcelona Spain (in press). [1964*b*]
6. Saxena. R. K., 2 A study of the generalized Laplace transform. *Proc. Nat. Inst. Sci. India*, Vol. 25-A, No. 6, p. 340-55. [1959]
7. Saxena, R. K., A study of the generalized Laplace, transform. 1, *Proc. Nat. Inst. Sci. India*, A 26, p. 400-13. [1960].
8. Saxena, R. K., Some theorems in Operational Calculus and infinite integrals involving Bessel functions and G-Functions. *Proc. Nat. Inst. Sci. India*, Vol 27A, No. 1, p 38-61, [1961].
9. Sharma, K. C., Thesis approved for Ph. D. degree, Raj. Uni. [1962]